Teaching Addition and Subtraction by the Method of Bidirectional Translation: an Empirical Study

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ABSTRACT

Bidirectional Translation, devised by the author, is a structured approach to the teaching of addition and subtraction which aims to give children greater understanding of arithmetical operations. The approach systematically involves both:

the translation of numerical representations into hypothetical, real world contexts;

and

the extraction of the appropriate numerical operations from hypothetical, real world contexts.

It is this emphasis on translation from and to both the numerical representation and realistic contexts which gives rise to the name, Bidirectional Translation.

An experimental group of 90 primary one children were taught to add and subtract (within 10) by the method of Bidirectional Translation. Post-test comparison of the experimental subjects' performance with that of a control group showed significantly superior performance on the part of the experimental subjects in terms of the utilizability of addition, the evocability of addition, the utilizability of subtraction and the evocability of subtraction for five different classes of verbal context, namely: Part-Part Whole, Separating, Joining, Equalizing and Comparison contexts. In all instances the probability of the

results being chance ones were less than 5% and in most, were less than 1%.

In both the experimental and control groups, most children performed better when they were required to utilize concepts than when they were required to evoke concepts. Similarly they performed better when they were required to add than when they were required to subtract. The differences, however, were not always significant.

It is suggested that the effectiveness of the methodology of Bidirectional Translation is rooted in a structure which allows the child to make his/her thinking explicit and which allows the teacher to monitor this.

CONTENTS

It may be, that in a thesis of this size, the reader does not wish to read everything or, indeed, to read the text in the order in which it is written. There follows, therfore, a summary of what is in each of the ten chapters.

Chapter 1 PRIMARY SCHOOL MATHEMATICS AS A PROBLEM AREA is introductory in nature and considers why mathematics should have a place in the school curriculum together with some examination of why mathematics is generally perceived as 'difficult'.

Chapter 2 THE PROCESS PROBLEM AND THE TRANSLATION PROBLEM is an attempt to clarify what is meant by problem solving per se and how this pertains to mathematics. It is argued that what has traditionally been seen as mathematical problem solving in schools is TRANSLATION rather than PROCESS problem solving.

Chapter 3 MATHEMATICAL CONCEPTUALIZATION tries to grapple with what this unobservable process might be. In so doing, it introduces and tries to make connections between some psychological concepts which have been described in the literature, namely: evocability, utilizability, intuitive intelligence, reflective intelligence and metacognition.

Chapter 4 THE CONSTRUCTIVIST VIEW is, as the title might suggest, a consideration of three Piagetian concepts, namely; classification, seriation and conservation, in terms of their implications for the teaching of mathematics in school. It suggests that Piagetian ideas may have little direct relevance for the teacher of mathematics.

Chapter 5 THE NEO-PIAGETIAN VIEW provides an account of mathematical learning which suggests that young children are far more capable than Piagetian theory gives them credit for. This alternative perspective gives primacy to counting.

Chapter 6 THE BEGINNINGS OF THE EMPIRICAL WORK explores the counting abilities of thirteen primary one children in the researcher's own class. The findings show that all of these children can count to 10 when there has been no formal input by the teacher.

Chapter 7 THE CONTINUATION OF THE EMPIRICAL WORK develops the researcher's investigations with her own class of primary one children by teaching them to add and subtract (within 10) using the method of Bidirectional Translation, an approach which she herself devised. The results are encouraging insofar as the children appear to understand what they are doing.

Chapter 8 THE EXTENSION OF THE EMPIRICAL WORK continues the work described in the previous chapter by testing the method of Bidirectional Translation experimentally. The design, subjects, stimuli, apparatus and procedure are all described.

Chapter 9 THE RESULTS, as suggested by the title, are tabulated, described and analysed. Broadly speaking the experimental subjects performed better than the control subjects though there are more detailed variations within this generalization.

Chapter 10 CONCLUSIONS FROM THE DATA draw together the findings of the previous three chapters in an attempt to explain the phenomenon under investigation, namely children's learning of the concepts of addition and subtraction. It is argued that because the approach to teaching these concepts appears meaningful to the children, Bidirectional Translation does not conflict with a Piagetian conception of children's learning.

PRIMARY SCHOOL MATHEMATICS AS A PROBLEM AREA

The Nature of Mathematics and its place in the Curriculum

Education, formal or otherwise, may be thought of as the process of enabling the individual to become an autonomous human being. The curriculum is that set of events which the individual experiences in the name of formal education. Many demands are made on the curriculum for the development of certain knowledge, certain skill, certain concepts and certain attitudes to have high priority. And what is deemed to be of high priority at one point in time may have little significance at another. In all of the debate surrounding curricular content, it is rarely, if ever, disputed that mathematics is important and should be a significant part of the curriculum. Why?

The simple, and simplistic, answer is that mathematics has utility. In our society some degree of mathematical skill is taken for granted, just as some degree of literacy is. Without skill in mathematics and reading a measure of independence is lost. However, a more penetrating response to the question of why mathematics is deemed important raises philosophical questions as to why this particular subject matter has to be defended as worthy of conveying to others.

Phenix (1964) argues that mathematics is a discipline in the **Symbolic Realm of Meaning.** By this he means that mathematics has a commonality with language insofar as language is a symbolic system for communication. However, mathematics differs from ordinary language insofar as:

1. it is an abstract means of communication with no necessary referents in reality;

2. its symbolism, designed to achieve complete precision in communication, is artificial.

From this it follows that what is abstract and artificial and neither concrete nor pragmatic is not going to be learned in some ad hoc, casual fashion. However, since, for Phenix (1964), mathematics is such a basic means of 'experiencing' meaning, it should be an essential part of the curriculum.

Hirst (1965) argues that there are certain discrete 'forms of knowledge', of which mathematics is one, which are central to all but the simplest kinds of human activity. One may not always, according to Hirst (1965), be fully aware of how influential these 'forms of knowledge' are on one's daily functioning but they do define and regulate our lives in the sense of extending and elaborating the meaning of human experience. The various 'forms of knowledge' (seven in all - mathematics, science, morals, aesthetics, religion, human sciences, history) are distinct insofar as each has its own system of interconnected concepts and each has its own validation procedures.

It is clear, even from this scant account of Phenix and Hirst, that mathematics may be seen as a distinctive way of making sense of the world.

The power of mathematics as a means of representing and communicating human experience has, in pedagogic terms, been interpreted variously but there are two discernible strands, for which the following descriptors are crude and incomplete. Historically, there has been the 'old' mathematics; conceived of more or less as a list of things to be 'done'. Having 'done' long multiplication of money, the teacher then proceeds to 'do' fractions or whatever may be next on the list, always emphasizing the computational processes as processes and paying scant attention to the interrelationships between processes. Then there has been the 'new' mathematics which aims for 'understanding' on the part of the child. To this end there has been a flood of structural material designed for use in the classroom. Through exploration of the material, the child allegedly perceives pattern and regularity, discovers procedures, and through investigation and experimentation develops a cohesive understanding of mathematics as a dynamic body of connected parts. As the former may be restrictive in what the teacher must 'do' and the child must learn, so the latter may be so fluid and nebulous as to defy any assessment of what has been learned.

From these very different views of what mathematics teaching means, there would appear to have emerged some sort of compromise. Mathematics, as the term is now typically used in Primary Education, refers to three distinct but not discrete components of Shape, Measurement and Number (Dept. of Ed. and Sc.,1979). Shape embraces the notions inherent in spatial relationships: height, length, width, symmetry, perspective, scale, two and three dimensional shapes and their properties, patterning and tessellations. Measurement means the quantification of a continuous amount, such as the weight of the sand, the length of the ribbon and so on. Implicit in the measurement process is the idea that we can only measure to a certain degree of accuracy; which depends on the measuring instruments used and the purpose behind the measuring activity. Over ontogenetic time, at least three stages in the development of an understanding of measurement can be distinguished:

1. comparison (ordering, equality, inequality, conservation, estimation, approximation);
2. use of arbitrary units (tinfuls, cubits, lentils, pendulum);
3. use of standard units, their notation and interpretation.

Number refers to the quantification of discrete amounts. This demands a computational precision based on sound conceptualization of counting, place-value, the four processes of number, integers, fractions, proportion and probability.

Clearly mathematics teaching is now conceived of as more than just computation. It is now seen as a part of one's general education, because even the most cursory reflection on what is a brief and incomplete taxonomy of content, draws attention to the importance of mathematical ideas in our daily lives. These ideas are basic to our understanding of, and competent functioning in, our environment as is exemplified in being able to tell the time, count our money, calculate the amount of curtain material needed, measure recipe ingredients, read timetables, and deal with charts, graphs and diagrams. This, however, is not to suggest that mathematics is merely a form of 'social arithmetic' for which mastery of the most basic skills is sufficient. While on the one hand, mathematical content (as just described) does have an immediate utility, what gives it that utility seems to be rooted in the mathematical morphology, as the following section will try to explicate.

Mathematics as a Means of Representation

Mathematics is not only a body of knowledge. It is a process. Mathematics makes extensive use of symbolic notation and many different situations can be expressed by the same mathematical statement. For example, the addition and subtraction relationships between the numbers 3, 4 and 7 (using them positively: 3+4=7, 4+3=7, 7-3=4 7-4=3) can, according to Carpenter & Moser (1982), be classified into at least six types of verbal contexts which in turn can describe potential real life situations:

Type 1 - Joining
Jean has 3 sweets. She buys 4 more. How many sweets does she have now? or Jean has 3 sweets. How many more does she need to have 7 sweets altogether? or Jean has some sweets. She bought 4 more. Now she has 7. How many sweets did she have to start with?

Type 2 - Separating
John has 7p. He gave 4p to Bob. How much money has he left? or John has 7p. He lost some money. Now he has 3p left. How much money did he lose? or John has some money. He gave 4p to Bob. Now he has 3p left. How much money did he have to start with?

Type 3 - Part-Part-Whole
There are 3 girls and 4 boys in the group. How many children are there in the group altogether? or There are 7 children in the group. 4 of them are boys. How many are girls?

Type 4 - Comparison
There are 3 girls and 7 boys in the group. How many more boys are there in the group? or There are 3 girls in the group. There are 4 more boys than girls in the group. How many boys are in the group? or There are 3 girls in the group. This is 4 less than the number of boys in the group. How many boys are in the group?

Type 5 - Equalizing - Add On
There are 3 boys and 7 girls in the group. How many boys should join the group so that there are the same number of boys and girls? or There were 3 boys in the group. 4 more boys joined the group. Now there are the same number of boys and girls in the group. How many girls are in the group? or 7 children each want a carton of milk. Mary gave out 3 more cartons of milk. How many cartons had she to put out to begin with?

Type 6 - Equalizing - Take Away
There are 3 cups and 7 saucers on the table. How many saucers should I take away so that there are the same number of cups and saucers on the table? or There were 7 knives on the table. I put 3 of them away so that there would be the same number of knives as forks. How many forks were on the table?

That different real life situations can be mathematically synonymous has important implications. Because the same mathematical statement can apply to more than one situation, results which have been obtained in one situation can be seen to apply to a different situation. Thus not only can mathematics be used to explain what has happened, it can be used to predict what will happen in a situation not yet experienced: as in how much petrol will be needed for a car journey not yet undertaken, or how much carpeting will be needed for the new house. This dual function of mathematics to explain and predict means that mathematics is an enormously powerful means of mental representation: that is, of how information is taken in, coded and remembered such that people behave on the basis of informed choice. As Lovell (1979) states, in mathematics lie "the origins of the concepts with which we structure the world".

The Disquiet about Mathematics

In spite of the central importance of mathematics in our lives, attitudes to mathematics appear to be negative. The authors of The 10 to 14 Report (The Consultative Committee on The Curriculum, 1986) point out that "many adults find little application for the mathematics of their schooldays in later life" and that "to claim to be 'no good' at mathematics is to invite sympathy rather than derision". Bell et al (1983) in their review of school children's attitudes to mathematics report that:

1. the utility of mathematics was not always readily perceived by pupils;

2. "throughout school a decline in attitudes to mathematics appears to go on".

As well as negative attitudes towards school mathematics, there is concern - as expressed in official publications and by the public at large - about the low levels of mathematical competence of many children both in primary and secondary schools. Brown (1979) cites two sources of evidence, Rees (1972) and Levy (1977), which suggest that there may be at least some justification for the criticism. Bell et al (1983) argue that if 'falling standards' is a fact, then the phenomenon owes as much to social and cultural factors as it does to the reality of mathematical attainment. Moreover, Bell et al (1983) point out that the 'falling standards' trend is world wide.

However, as McIntosh (1977) points out, the dissatisfaction with the mathematical understanding of the young has a long history. As far back as the turn of the century reports have been published which suggest that while children may be mechanically accurate in computation, they were unable to apply their skills in contextualized situations. McIntosh goes on to list the recommendations for improvement in mathematical understanding which were being suggested sixty to one hundred years ago! They were as follows:

1. Don't start formal work too early.
2. Use materials and start from practical activities.
3. Give children problems and freedom initially to find their own methods of solution.
4. Children must have particular examples from which to generalize.
5. Go for relevance and the involvement of the child.
6. Go for reasons and understanding of processes. Never give mechanical rules.
7. Emphasize and encourage discussion by children.
8. Follow understanding with practice and applications.

McIntosh's contention is that although there is not evidence of falling standards there is nevertheless a serious problem: while the above mentioned list of recommendations is almost universally agreed in principle, "their translation into accepted and practical terms for the majority of primary teachers has not yet come about".

The researcher's experience of 20 years of teaching children between the ages of 5 and 14 years would, broadly, agree with McIntosh's view that mathematics education has emphasized a 'skills-in-a-vacuum' approach. The methodological route for presenting any mathematical topic goes something as follows:

1. cursory reference to concrete material;

2. graded presentation of, and practice in, what Brown (1979) refers to as "algorithmic skills" with the 'bright' children completing more and more examples to keep them busy (this variously being referred to as reinforcement,

enrichment or extension exercises!), while the 'dull' children struggle to apply the formula(e) to a few 'simple' examples;

3. application of algorithmic skills to 'problems' involving numerical computation: this stage being tacitly recognized as the prerogative of the 'bright' children.

Nor is this account of mathematics teaching idiosyncratic. Skemp (1971) claims that:

> What is inflicted on all too many children and older students is the manipulation of symbols having little or no meaning attached, according to a number of rote-memorized rules.

This, in spite of official arguments that the "main reason for teaching mathematics is its importance in the analysis and communication of information and ideas" and that "the mere manipulation of numerical or algebraic symbols is of secondary importance" (Department of Education and Science, 1985).

If this method of teaching were successful insofar as it resulted in people being mathematically competent and confident, the end might justify the means. But the difficulty created by this method of teaching is clearly delineated by Skemp (1971):

> Learning to manipulate symbols in such a way as to obtain the approved answer may be very hard to distinguish, in its early stages, from conceptual learning. The learner cannot distinguish between the two if he has no understanding of mathematics. And all the teacher can see (or hear) are the symbols. Not being a thought reader, he has no direct knowledge of whether or not the right concepts, or any at all, are attached. The way to find out

> is to test the adaptability of the learner to new, though mathematically related, situations. Mechanical computation does not do this.

Thus it would appear that there are two distinct facets to mathematics: on the one hand there is the manipulation of symbols in approved and recognized ways, and on the other there is conceptual learning. Most teaching attention appears to be addressed to the former. Even with the advent of SPMG Infant and Primary Mathematics (1981) and of Ginn Mathematics (1983), the two most recent innovations in Scotland, maths education in the primary school seems to be based on the premise that "if a child repeats a meaningless process enough times it will become meaningful" (Holt, 1964). This is not to say that teachers are necessarily satisfied with this approach, nor that they should be, but, given the constraints of time; of policy at school, regional and national level; and the emphasis on providing a broad and balanced curriculum in the primary school; it is as much as many teachers can do to try to implement the schemes or programmes of work which they may feel have been foisted upon them.

The Problem

The problem with the 'skills-in-a-vacuum' approach to maths education seems to be rooted in two questionable assumptions: firstly, the assumption that there is a necessary, unidirectional progression from using concrete material through learning algorithmic skills to applying the skills in so called problems and, by extension, the second assumption, that conceptual understanding of, and computational mastery in, a mathematical task are one and the same thing.

Dealing with these assumptions in reverse order, what does it mean to have conceptual understanding of and computational mastery in a mathematical task?

Conceptual understanding refers to a network of meaningful relationships which underpin the performance of skills. Conceptual understanding is exemplified by the ability to remedy a memory failure and go back to 'first principles' and/or adapt a skill or procedure to a new situation. As Bell et al (1983) point out:

> The real importance of the conceptual structure is that as a richly inter-connected network it constitutes a stable memory structure, in which any particular link which fades is relatively easily reinstated.

They go on to say that:

> The learning of a new concept or relationship implies the addition of a node or link to the existing cognitive structure, thus making the whole, if anything, more stable than before.

Computational mastery, on the other hand, refers to sets of useful tools in the form of standard procedures. These tools are the skills which have been distilled out of a variety of diverse strategies. They are the final, shorthand versions which mathematicians through their extensive experience have deemed useful ways of attacking the calculations required in mathematical tasks. To be able to perform these skills requires remembering what to do first, next, last and so on in any given mathematical operation. As Bell et al (1983) point out, "the learning of a skill requires the establishing of a set of more or less arbitrary links between the steps".

Relatively recent research by Rees (1972) and Brown (1979) has shown that the identification of the mathematical operation (that is, having a conceptual understanding of what is required) and its computation (that is, of being able to effect the algorithmic skill) have a fairly low positive correlation. In other words, being able to perform the algorithm need not imply any conceptual understanding of what one is doing and conversely, understanding does not guarantee computational proficiency. Similarly, Begle (1979) concludes that improvements in computation and in higher level acquisitions (comprehension, application and analysis) develop relatively independently of each other, with not very much interaction, and that "computation achievement is something quite different from achievements at higher cognitive levels".

Since conceptual understanding and computational mastery are not one and the same thing, it now remains to turn to the first assumption, that there is a necessary, unidirectional order from concrete experience through repetitive practice to the application of mathematical skills in potentially real situations.

In using concrete material, one is attempting to illuminate what otherwise might be too abstract for comprehension. It was Bruner (1966) who said that "any domain of knowledge can be represented in three ways: enactive representation (actions), iconic representation (pictures) and symbolic representation (symbols)". While Bruner pointed out that the younger the child the more likely he/she would be to use enactive representation and later progress through iconic to symbolic representation, he did also stress that "actions, pictures and symbols vary in difficulty and utility for people of different ages, different backgrounds, different styles". Thus concretization of a thinking task is a strategy which can facilitate the thinking process. It is not something that must immutably be regarded as a pre-requisite to further, sophisticated thought but should, rather, be regarded as an integral part of thinking which may at times be useful.

Empirical support for this argument can be found in at least one source. The classic Wason four-card-selection task (Wason & Shapiro, 1971) requires subjects to solve the problem of naming those cards and only those cards which need to be turned over to determine whether the rule is true or false. When presented in the abstract form of 'if a card has a vowel on one side, then it has an even number on the other', subjects frequently cannot solve the problem but when the task is presented in the concretized version of 'if I go to Manchester, then I travel by train'; then the majority of subjects can solve the problem. More interestingly, however, is the finding of Johnson-Laird et al (1972) that practice on a concretized version does not transfer to subsequent abstract presentation.

While for Piaget (whose work will be considered in subsequent chapters) intellectual growth consisted of a series of 'stages' through which the individual has to pass before being capable of mature thought, for Bruner et al (1966) the growth of the human intellect is a successive mastering of the three forms of representation. However, for neither Bruner nor Piaget is there the suggestion, not even the implication, that as the individual takes on greater sophistication in thinking he/she never 'regresses' to less sophisticated modes of thought. Both acknowledge that the typically 'mature' thinker will, in situations for which more 'immature' modes of thought have not been fully worked through, revert to less 'mature' modes of thought when abstract and symbolic forms do not serve the purposes of the individual. The 'mature' thinker who does not concretize a thinking task does not do so because he/she does not need to. The 'mature' thinker who, on the other hand, will, in certain situations, concretize a thinking task, does so because he/she needs the **support** of concretization.

The utility of concretization, then, is as an aid to thinking and reasoning. As an end in itself, the use of concrete material probably has little value. If, in using concrete material, there is no attempt to strip the 'noise' from the activity and extract the underlying 'mental meaning', it seems unlikely that the concretization has been of much benefit. It follows, therefore, that rather than view concretization as a necessary pre-requisite to thinking, it would be more appropriate to regard concretization as a prop to thinking, a prop which becomes less and less important with increasing maturity in thought but which may nevertheless be called upon from time to time.

Application or Abstraction?

If we can argue that the traditional importance placed on concrete materials (both in terms of their intrinsic value and in terms of temporal positioning) is largely mythical, can the same be said of the emphasis placed on algorithmic skill practice followed by the application of such skills in problem form? It is certainly an assumption (albeit tacitly held) that "verbal problems are difficult for children of all ages" (Carpenter & Moser, 1982), and that children must learn the necessary operations "before they can solve even simple verbal problems" (Carpenter & Moser, 1982).

But someone who lacks the skill to compute the division operation suggested by the task, 'John wants to share 9 sweets equally amongst himself, Tom and Harry, so how many sweets will each boy get?' may nevertheless be able to carry out the task. The task could be correctly effected by sharing out: by constructing a one-to-one correspondence between boys and sweets or, at a more advanced level, by repeated subtraction. Very young children can, and do, solve simple verbal problems using their own invented/developed procedures (Carpenter & Moser, 1982; Starkey & Gelman, 1982). This would suggest that, contrary to folklore, children do not learn maths and then apply it but rather, from their experiences of applications in the real world they abstract the mathematical properties from the applications. Clearly such a proposition is too simplistic to account for the learning of 'pure' and advanced mathematics, but for the initial introduction to the formalisms of this means of communication, it does have an appeal.

The intuitive appeal for the view that young children may abstract rather than apply their mathematical knowledge gains some support from the following analysis.

The experienced, practising teacher will in all likelihood have noticed children's mathematical behaviour, when the children are required to solve verbal problems. Something approaching one or more of the following patterns may have been observed. First there are the children who never really become engaged in the task. They state that they don't understand, don't know what to do, don't <u>want</u> to explore the task. They may try to ask the teacher or somebody else what the solution/answer is. This is the case of children, as they themselves have clearly indicated, not being able to conceptualize what is required, within the constraints of the given task; in which case the children are being asked to engage in a task which, to them, is meaningless when, instead, they should be engaged in some kind(s) of work which is a precursor to the task in hand. For these children, the application of mathematics does not seem a viable proposition!

Next there are the children who engage in the task using what the teacher might regard as an 'immature' approach (even when these children have allegedly been taught more sophisticated algorithms), such as carrying out a division task by constructing a one-to-one correspondence. For such children it is vitally important that they be allowed to become confident in carrying out tasks by this fashion. When thoroughly secure in their own methods, the possibility of the task being effected more economically by an alternative method, such as, say, repeated subtraction, can be suggested and/or demonstrated. And finally, when the children begin to make the transformation for themselves into symbolic representation, the algorithmic skills can be explicated for potential application. For such children their initially 'immature' approach is a function of their understanding of the concept of division and until such times as they can appreciate the utility of the more conventional solution procedures, it cannot be said that they are applying what they have been taught (but which, it seems, they have not learned!).

Finally, there are the children who engage in the task and isomorphs of the task with confidence, using the most economic and appropriate algorithmic skills. Such children have developed a 'higher-order principle': a combination of understanding and rules which are stored in memory and which can be retrieved to address similar tasks in a quick, routine way. While these children are applying their knowledge, it is not at all clear that the application **follows** the abstraction. It just appears that way now that application and abstraction have become integrated with each other.

The children who exhibit the third type of behaviour are a 'pleasure' to teach. They probably don't have much need of the teacher anyhow! The children who exhibit the first type of behaviour are viewed by the teacher as having problems and in need of remedial help which the teacher has no time to give them!

But it is the children who are exhibiting the second type of behaviour who are most interesting from the point of view of developing the children's mathematical thinking. These children are making very clear to the teacher their level of immaturity or sophistication in mathematical thinking. This unwitting exposure by the children is the very phenomenon which will allow the sensitive teacher to facilitate the children's development and one which teachers should be at pains to nurture.

From the above discussion, it seems that the assumption that algorithmic skills practice necessarily precedes their application is false. The obverse suggestion, that skills practice should follow experience in verbal problems, is just as unsatisfactorily extreme: one cannot apply skills which one has not learned. The possibility emerges that there should be some sort of compromise, not in the sense of reducing our expectations of pupils (since many would argue that these are already too low), but in the sense of strengthening the links between the algorithmic skills and the mathematical understanding. The relationship between the two should not be viewed as unidirectional, but rather as bidirectional.

The content of this first chapter has, perhaps, painted a somewhat depressing picture of the current state of mathematical achievement. If this phenomenon of low mathematical achievement were easy to understand, then it would presumably be possible to attenuate the worst effects of this. However, the issues involved are complex. The remainder of this thesis is an attempt to analyse what the issues might be and further, an attempt to address them. Chapter two considers the nature of 'problems' and 'problem solving' in relation to school mathematics.

In summary:

(i) though mathematics is an important constituent in our competent functioning in the world, mathematics teaching is viewed, retrospectively, as being of little value, by many people;

(ii) children have, historically, found it difficult to apply computational skills in contextualized situations;

(iii) this may be a function of teaching methodologies which subscribe to limited behavioural objectives and aim at 'transmission' models of learning;

(iv) conventional assumptions underlying mathematical learning are now held to be questionable and thus the time is ripe for revising such assumptions in the light of alternative teaching strategies.

THE PROCESS PROBLEM AND THE TRANSLATION PROBLEM

The researcher's interest in formal arithmetical competencies in children, springs from her sympathy with the literature (Brown, 1979; Cockcroft, 1982; Bell et al, 1983; Hughes, 1983; Hughes, 1986) in which it is argued that children see little if any relationship between arithmetical operations and their applications in the real world; and from her extensive experience in primary education where the children's lack of understanding of the significance and/or utility of mathematics was, sadly, evidenced on an almost daily basis. Essentially, the point of concern is that, crudely put, children may know <u>how</u> to compute but this does not ensure that they know <u>when</u> to compute.

This phenomenon of relating the <u>how</u> and the <u>when</u> in mathematics is frequently referred to as problem solving. As Polya (1981) points out, mathematical "know-how" can be thought of as the ability to solve problems. The general consensus of alarm expressed by many researchers (Ballew & Cunningham, 1981; Threadgill-Sowder & Sowder, 1982; Wollman, 1983; Moyer et al, 1984; Fischbein et al, 1985) about children's poor problem solving performance is neatly summed up in the following quotation by Carpenter et al (1980):

> If it were necessary to single out one area that demands urgent attention, it would clearly be problem solving. At all age levels, and in virtually every content area,

> performance was extremely low on exercises requiring problem solving or application of mathematical skills.

It is, however, the present author's contention that great care should be taken in using the terms, PROBLEM and PROBLEM SOLVING; which should not be conflated with the term, APPLICATION OF MATHEMATICAL SKILLS. The remainder of this chapter then, is concerned to make the distinction between the two clear, by arguing through negative example that problem solving is much more than the application of mathematical skills, but that the application of mathematical skills is, in itself, of enormous importance. This strategy may seem clumsy and laborious but it is felt necessary, by this author, to make the distinction between problem solving and application of mathematical skills absolutely clear.

Firstly, what is meant by the terms, PROBLEM and PROBLEM SOLVING?

A problem is a hindrance, a blockage which prevents us from easily or immediately realizing an objective. Newell & Simon (1972) define a problem as a situation in which a person wants something (the goal state) but does not know immediately how to achieve the goal state, given his/her present conditions (the initial state).

Problems are pervasive and various. They can be:

large - how to deal with drug abuse;

small - how to locate one's misplaced spectacles;

short range - how to get into the car when the keys are locked inside;

long range - how a spinal injury victim can learn to walk again;

well defined and specific - how to calculate how much wallpaper is needed for the sitting room;

general and poorly defined - how to teach children to become mathematically effective.

Problems are thus to do with the actuality of our being. They are part of our individual and corporate reality and manifest themselves in the context of a person's or people's cognitive and affective constructions of reality. Simplistically put, this means that if one does not construe a given situation as being problematic - either through lack of interest or lack of perception - then there is no problem for that person! Equally, what is a problem for a person today may not be a problem tomorrow because one's perceptions of the same set of conditions have changed. More than that, however, if one knows immediately how to proceed and how to effect a solution when faced with a set of conditions, one is not faced with a problem. Instead, one is faced with an exercise, a task or even just a chore. Yet it may be that that exercise, task or chore was once a problem given that when confronted with a set of conditions for the first time we will, to a greater or lesser extent, be novices in how best to proceed. The inter-dependence of problem and problem solving is reflected in the terms being used interchangeably. Problem solving is the need to do something about what is perceived as a problem. Conversely if there is no problem there need be no problem solving activity.

As has been implied above, problem solving is goal directed behaviour to effect a solution to what is seen as personally challenging to the individual. The individual's engagement with the problem, therefore, requires the person to make judgements, to reason, to understand, to remember, to pay attention; all of which can be described as **thinking processes.** Such thinking processes are active (in the sense that one has to do them personally - they cannot be done for one), exploratory (in the sense that at any point they can be abandoned - all decisions are tentative and subject to revision), and experienced-based (in the sense that one probably does not think about things one cannot conceive of). Thinking is a cognate human activity. It is reasonable, therefore, that problem solving or thinking should feature large in the formal education of the individual.

But what are the implications of such claims for mathematics teaching?

One implication is that for the learner a range of choice and discretion over the mathematical tasks he/she engages in might be helpful. The teacher, quite understandably, need not necessarily be able to determine what will, or will not, truly engage the learner. Nevertheless, the teacher must manage the children's learning such that, as Burton (1980) points out;

> The mathematical task must pose a question which is intriguing or meaningful to children of the relevant age. The question need not always be real in the sense of being environmentally based since it has been found that children get greatly involved and achieve high satisfaction from cognitive challenges of the puzzle variety. The important factor is that the question is so posed that there is the chance for it to become their own.

Another implication is that as a result of the task the learner's thinking should have been facilitated. In some way or ways the learner should 'know' more at the end of the task than he/she did at the beginning. This 'knowledge' is not so much of the declarative variety of 'knowing that' as of the procedural variety of 'knowing how', such that the learner is less dependent than previously on rigid algorithms and restrictive heuristics as strategies for problem solution. For Burton (1980), this will happen provided the task "calls out a range of explicit problem solving skills and procedures which are reinforced during the problem solving process".

According to Burton (1980), the skills are to do with "comprehension, transformation and communication". These skills are divided into two categories, representational skills and information analysis skills.

> Representational skills are those which facilitate the construction and/or use of different modes of presentation and the appropriate choice of mode:
> 1. Linguistic;
> 2. Pictorial;
> 3. Concrete;
> 4. Symbolic;
> 5. Translation;
> 6. Modelling.
>
> Information analysis skills are those of collecting, organizing, analysing and presenting information:
> 1. Using representational skills to identify data and information;
> 2. Making knowns and unknowns explicit;
> 3. Using systematic arrangement of information;
> 4. Presenting data.

Burton's procedures for problem solving are the methods of tackling a problem such that the problem is moved into the domain where skills can be applied. The procedures are seen as being dynamic and are divided into three categories:

> Entry procedures are those which enable the solver to get to grips with the problem. They are the mechanisms which expose the problem and make it amenable to attack. They include techniques such as trial and error, defining of terms and relationships, information ordering and so on;
> Attack procedures move the problem towards solution, although not always successfully. They include techniques such as working backwards, trying related problems, trying special cases, using empirical argument and the systematic control of variables;
> Extension procedures attempt to answer the question, 'where do we go from here?'. They increase the solver's understanding of the problem and help him to place the problem in a known context or to develop understanding of a new context. Extension procedures include generalisation to a class of problems, finding isomorphic problems and creating new problems.

Burton (1980) suggests that problem solving is a means by which mathematics can be pursued. At the same time she points out that the skills and procedures which are required to solve problems are not the exclusive preserve of mathematics but are appropriate to "other experimental activities". This type of activity emphasises the thinking **processes** for obtaining solutions. It demonstrates the need for, and gives practice in, understanding problems, developing and carrying out solution strategies, and evaluating outcomes. As such these problems are what Charles & Lester (1984) would describe as **PROCESS PROBLEMS.**

If PROBLEM SOLVING is a **process** and the PROBLEMS to be solved are **process problems**, how does this tie up with typical problems in mathematical textbooks? Let us consider the following examples:

> <u>example 1</u> Marty saw 59 old cars and 38 new cars. How many did he see altogether? (Ginn Mathematics, 1983);
> <u>example 2</u> In a 500 kilometre stock car race all 17 starters were able to finish. How many kilometres were driven in all? (Ginn Mathematics, 1983);
> <u>example 3</u> There are 12 classes in Marshall Primary School. Each class has 34 pupils. If 37 pupils are absent, how many pupils are present? (Ginn Mathematics, 1983);
> <u>example 4</u> How many lemonade bottles, each containing 1.55 litres, can be filled from a tank which holds 372 litres? (S.P.M.G., Heinemann, 1981).

It was suggested earlier that, for the learner, the element of choice might be facilitative. In the typical situation, in schools, the only person likely to be making the choice as to whether or not the learner would be addressing any of the above four examples would be the teacher! It was further argued that problem solving enabled learning by extending comprehension and communication. It is not at all clear that, in the four examples above, communication and comprehension are being extended particularly if we apply some intuitive analysis of what the child is possibly doing.

These examples came from children's textbooks, therefore it is reasonable to assume that the children would be required to **read** the problems for themselves. There is nothing wrong with making reading demands of the learner provided he/she **can** read. Reading involves more than being able to decode words. It means the skill of extracting meaning from words written on a page, the competent performance of which involves bringing to the text what knowledge one already has of a topic (Goodman, 1967; Ryan & Semmel, 1969). If the learner's reading skill is not commensurate with the structural/lexical/contextual complexity of the text, then the learner cannot begin to think mathematically, but this does not mean that the learner could not address the problem if it were presented in some other medium. In other words, what is being argued here, is that in textbook type mathematical problems the variable of reading may be given a place of undue importance.

A second demand being placed on the learner is that he/she has to translate the linguistic expression into mathematical symbolic notation. This means having some sort of conceptual awareness of the task. Conceptual awareness is shown by the ability to remedy a memory failure and go back to first principles and/or adapt a skill or procedure to a new situation. In the developing child, the concept will be available to a greater or lesser extent. For example he/she might correctly solve the problem, 'Jim has 3 marbles and his friend gave him 4 more. How many marbles does Jim have now?' but not be able to solve example 1 cited above because, as yet, the learner's understanding can only be applied to small numbers (Flavell, 1971; Gelman & Gallistel, 1978; Hughes, 1986). This raises the possibility that the learner cannot make the problem his/her 'own', cannot mentally represent the problem to him/her self, in which case he/she does not understand what is required and therefore cannot begin to proceed. In other words, what is being argued here is that textbook type mathematical problems may be too restrictive, so that instead of opening up possibilities for exploration and investigation the stimulus material 'blocks' the child.

A third demand made of the learner is that he/she should be able to activate the appropriate algorithm and perform the necessary computation correctly. This is, after all, the purpose of the textbook type mathematical problems - to identify what is required and do it!

Textbook type mathematical problems are then of a different order altogether from PROCESS PROBLEMS. Textbook type mathematical problems require the child to identify and accurately effect the relevant algorithm(s). These problems are what Charles & Lester (1984) would describe as **TRANSLATION PROBLEMS** because they involve translating the given information into numerical notation. Additionally, these **translation problems** can be thought of as simple (the familiar one step problem that can be solved by adding, subtracting, multiplying or dividing as in examples 1, 2 and 4) or complex (multiple step problems requiring two or more operations to find the solution as in multiplying and then subtracting in example 3).

Such **translation problems** as are typically found in mathematics texts, and are often referred to as problems, do not then involve problem solving in the sense of **process problem** solving described above. This study is not concerned with investigating the **process problem**. It is, however, very much concerned with the **translation problem**. This is not to say, however, that the **process** and **translation problems** are mutually exclusive. One way of describing the relationship is to use the analogy of Gagne's (1977) learning hierarchy. This is a system of increasingly more complex learning processes, in which lower levels of learning are prerequisite to higher levels. From the 'top' the levels of learning are:

Problem Solving
|
Principles
|
Concepts
|
Discriminations
|
Stimulus-Response Connections

Accordingly, what constitutes a **translation problem** is classed as being at the conceptual level. This is subordinate to, and a necessary precursor of, problem solving per se, "a process by which the learner combines previously learned elements of knowledge, rules, techniques, skills and concepts to provide a solution to a novel situation" (Orton, 1987). In other words, the **process problem** subsumes the **translation problem.**

The **translation problem** is, in effect, the "application of mathematical skills" (Carpenter et al, 1980). The translation between a verbally described situation and the appropriate symbols is an important part of **process problem** solving. However, such translation is also an incomplete characterization of **process problem** solving. Nevertheless, the translation is of fundamental importance in the understanding of mathematical operations. Without the ability to make the translation, there is no conceptual understanding of the operation in question. As Vergnaud (1982) points out:

> Concepts and symbols are two sides of the same coin and one should always take care to view students' use of symbols in the light of their use of concepts. In other words the ability to solve problems in natural language issued from ordinary social, technical or economic life is the best criterion of the acquisition of concepts. Reciprocally, it is essential to know how mathematical symbolization helps students.

Typically, the failure to successfully perform a **translation problem** is attributed to reading deficiency. This reading deficiency may be a generalized, cross-curricular one or it may be one arising out of what Kane (1967) calls Mathematical English, "a hybrid language composed of ordinary English commingled with various brands of highly stylized formal symbol systems". Kane (1970) believes that:

> Mathematical English and ordinary English are sufficiently dissimilar that they require different skills and knowledge on the part of the readers to achieve appropriate levels of reading comprehension.

Because, for whatever reason, the child cannot 'read' the problem but can 'do' the arithmetic, he/she may be allowed as Glenn (1978) points out, to complete page after page of 'sums' omitting those parts which use number in a verbal context. The result is that the child "may never learn to relate the symbols effectively to situations involving number" (Glenn, 1978) and may therefore "not develop a sense of number" (Glenn, 1978). Unwittingly the teacher is widening the gap between numerical representation and its possible real world application when conceptual understanding of an arithmetical operation requires that the two be closely linked if not intermeshed.

If, for example, a child can perform the computation, 9 divided by 3 when it is presented as 9÷3 but cannot find the answer when the same operation is presented as 'Mary shares 9 sweets among her three brothers. How many sweets does each boy get?' then such a child does not even have the most rudimentary concept of division (although he/she may have some computational skill in the division process) because having a concept means being able to apply associated skills in new situations or, to put it another way, being able to generalize from the learned situation to new contexts. Leaving aside the issue of whether or not the child can read the **translation problem**, the child's ability to address the **translation problem** in any purposeful way seems to be crucially dependent on his/her having the appropriate concept as part of his/her cognitive structure. The child who can 'do' the **translation problem** has some sort of conceptual understanding of the mathematical content, whilst the child who cannot (and there are many of these, as teachers know intuitively and research has substantiated) would appear to be being asked to do something for which he/she is not equipped.

This failure to conceptualize the mathematical content is not restricted to very young children. Brown (1979) for example, found that one third of a representative sample of 1089 children aged 11 to 12 did not recognize that a multiplication algorithm was required in the problem, 'An oven tray for cooking little cakes will hold 56 cakes. A baker fills 28 trays. How many cakes will he cook?' She also found that when children were asked to interpret the notation in terms of a verbal context, less than one third of them could think of any practical problem they could solve by multiplying 56 and 28.

Clearly, some type of teacher intervention is required to enable children to make the links between computation and its application. The lack of skill application is further evidenced in Hughes's (1986) work where he found that while small children can very ingeniously represent quantity, "few managed to represent addition and subtraction" and even then, the "few" made no effective use of conventional operator signs, in spite of the fact that "they were using the formal symbolism of arithmetic every day in their workbooks". For example, although children were experienced in completing operations such as 5-3= or 2+4=, when they were asked to show the 'dynamic' nature of addition and subtraction, to show that an initial state had been changed, as in 'First we had two bricks and then we added two more', the most common response was to show the final quantity alone without reference to the action that had been carried out.

It cannot be conceded, however, that all types of teacher intervention (whatever they may be) are enabling to the child. Carpenter et al (1981) argue that before children have had "formal instruction" in addition and subtraction they have a high rate of success in solving addition and subtraction verbal problems. They conclude:

> Very few of them used the wrong operations in their solutions. Since this error has been observed primarily with older children who have already experienced formal instruction in addition and subtraction, it may actually be a result of learning symbolic representations. Because the operations are initially learned outside the context of verbal problems, they have no basis for using their natural intuition to relate the problem structure to the operations they have learned. In other words, their natural analytic problem-solving skills are bypassed, and they too often resort to relying on superficial problem characteristics to identify the correct operations. This may result not only in a superficial concept of addition and subtraction but also a decline in general problem solving.

Perhaps, however, teacher intervention could be based on what children are actually doing. Resnick & Ford (1984) suggest that if only teachers would "cultivate their own skills of observing and questioning" then they would ultimately "begin to note details of children's thinking that had not been apparent before and find themselves able to follow children's lines of reasoning more clearly". What to the adult may seem an obvious arithmetical algorithm couched in a verbal context, may be perceived differently by the children. For example, Brown (1979) found that although some children could not recall the 'official' algorithm for subtraction they could nevertheless solve practical problems quickly and efficiently using their own private procedures which resulted from "a good conceptual understanding of the operation involved". Similarly, Carpenter & Moser (1982) found that young children made far greater use of counting strategies than they did of number facts when addressing addition and subtraction **translation problems.**

The phenomenon of the arithmetical **translation problem,** then, is well documented, and the general conclusion is that performance on **translation problems** is poor. Such a conclusion is worrying since the (arithmetical) **translation problem** is synonymous with (arithmetical) conceptual understanding: successful performance on the **translation problem** means that the concept(s) involved in the problem's solution is/are being established in the conceptual system. Even more worrying, however, is the possibility that poor performance on **translation problems** may have implications for conceptual development in general. Bryant (1985) argues that the distinction between knowing <u>when</u> and knowing <u>how</u> to perform an arithmetical operation "might be one instance of a rather general rule in children's cognitive development". He points out that many of the erstwhile conclusions about young children's apparent incapacities were drawn from inadequate evidence: just because the child does not display a particular behaviour does not mean that he/she <u>cannot</u> display that behaviour. It merely means that, for whatever reason(s), the situation was not 'conducive' to the manifestation of the particular behaviour. If children are not using the 'cognitive tools' they possess when it would be beneficial to them to do so, their development is not, it would seem, being facilitated. Bryant (1985) argues that there is;

> incontrovertible evidence that children's performance on most cognitive tasks depends on two quite separable things. One is the possession of the skill needed to solve the task and the other is their recognition of when that skill is needed.

This in turn raises questions as to where in the mathematical situation the child's difficulty lies: is it in the recognition of what calculation is needed? is it in actually carrying out the calculation? or is it in both?

Thus it would appear that not only is it mathematically desirable, it is psychologically necessary (from the point of promoting cognitive development) that the child be able to perform the **mathematical translation problem** successfully.

This chapter has been at pains to give the **translation problem**, otherwise known as the application of mathematical skills, its due and proper place; while at the same time distinguishing it from the **process problem**. The mathematical **translation problem** is intimately bound up with mathematical conceptualization. The next chapter analyses what mathematical conceptualization means.

In summary:

(i) it has been argued that the terms, problem and problem solving are, generally speaking, used with a lack of precision in meaning as they pertain to mathematics;

(ii) more precisely, problem solving, according to Gagne (1977) and Burton (1980), requires what we casually call 'thinking', is dependent on a large store of knowledge and capabilities, and is not restricted to mathematical content alone;

(iii) those tasks which require the identification and execution of an arithmetical representation in the context of a hypothetical, real world scenario are more appropriately referred to as mathematical **translation problems** according to Charles & Lester (1984);

(iv) empirical evidence from Brown (1979); Carpenter et al (1981); Bryant (1985) suggests that the successful execution of the **translation problem** is a criterion of conceptual understanding of the arithmetical operations involved;

(v) for Bryant (1985) successful performance on **translation problems** may be important for cognitive development per se.

The Abstract Nature of Mathematics

All established areas of knowledge or forms of understanding (Dearden, 1968) have their own explanatory concepts and related means of verification. Mathematics is one such domain of knowledge. These concepts, the result of hard thinking on the part of previously successive generations of people, are available to subsequently new generations of people. The question is, how are such concepts made available to people? Clearly, all people do not (fully) avail themselves of mathematical concepts, otherwise there would not be the concern, as exists today, for the impoverished nature of mathematical learning. And true mathematical learning, it seems, is heavily dependent on learning concepts. Brown (1979) points out that "conceptual learning is obviously fundamental in mathematics". And Orton (1987) states that "mathematics learning consists very largely of building understanding of new concepts onto previously understood concepts".

Beyond the claim that "the actual construction of a conceptual system is something which each individual has to do for himself" (Skemp, 1971), the process of conceptualization in general and mathematical conceptualization in particular appears not to be clearly understood. One explanation of the conceptualization process (Bourne & Restle, 1959; Collins & Quillian, 1969; Meyer, 1970; Rips, 1975) is that the critical attributes of a concept (which in the case of the concept, insect, would be the number of body segments and legs) are abstracted from a variety of positive and negative exemplars of the concept and constitute a list of defining features against which new instances are compared. This explanation is, however, problematic. Firstly it is difficult to accept that feature comparison is always and necessarily involved in deciding whether class membership obtains. Intuitively, we class apples as fruit, cats as animals and daffodils as flowers holistically, without recourse to feature analysis and feature matching. Secondly, and following on from the first point, it is questionable to suggest or imply that defining features can be supplied for all concepts. What are the defining features of a house, for example?

Another explanation (Rosch, 1975; Rosch & Mervis, 1975; Rosch et al, 1976) is that we construct a prototype, or the best example of a category (which has most of the attributes that are common to most members of the category and fewest of the attributes of nonmembers of the category). By unconsciously calculating the frequency of feature representation we compare new instances of a category with the prototype and decide whether or not the instance is to be included in the category. Thus in British culture, robins and sparrows would be very prototypical of the concept, bird, while ostriches and penguins would be non prototypical. But this explanation is not without its problems either. While the prototype model avoids the feature comparison paradox of classifying an instance according to a list of defining features which are unknown or unknowable, it nevertheless does not allow for clear specification in ambiguous cases. By class membership being decided on the criterion of 'best fit', it follows that when an instance falls into an area where boundary categories overlap, classification will be context dependent, with resulting possibilities of disagreement. A pistol, for example, might be an ornament in one context and a weapon in another.

Yet another explanation (Brooks, 1978) is that instances of a category are collected on the basis of their overall, global similarity to an earlier, known instance. This explanation posits concept acquisition as being more intuitive, implicit and non analytical, with far less conscious hypothesizing or analysis; and, what is more, could account for the way in which children acquire concepts insofar as complex feature combination rules, the encountering of many instances, and a perfect memory of previous experience are not necessary conditions.

The empirical evidence for each of the above explanations is considerable, so rather than posit one in favour of the other two it is perhaps wiser to consider each as a plausible model, depending on the individual learner and other context variables. It would not then be unreasonable to conclude that these alternative explanations can co-exist.

The process of conceptualization is perhaps not well understood because the concept of a concept is in itself elusive. A lexical definition of the term, concept, such as will be found in any reputable dictionary, indicates that a concept is a general notion. This is hardly illuminating. Furthermore, usage of the term, concept is not necessarily constant among those practitioners who might be credited with knowing what concepts are. For example, Rae & McPhillimy (1985) claim that "after teaching a concept, the teacher will wish to check that the child has indeed acquired the concept". This suggests that a concept is an item which can be directly imparted by the teacher and directly received by the learner; that it is a discrete, tidy, unambiguous entity. On the other hand, Babin (1980) claims that "the teacher does not 'give' a concept to the learner: students acquire concepts through appropriate learning materials and experience". This suggests that a concept is a node or 'building block' (in the knowledge system) which is dependent on other nodes or 'building blocks' for its existence and which is built up piece by piece out of whatever sense the learner makes of his learning environment. Moreover, Babin's view implies an active process of constructing meaning;

meaning which can be revised, updated and enlarged as a result of learning experiences.

However, the apparently opposing views of Rae & McPhillimy (1985) and Babin (1980) may not be as mutually exclusive as a rapid perusal of them might suggest. From what has been said above, one can speculate that a concept is a <u>mental</u> structure. As such a concept has no physical basis in existence. It refers to some underlying competence. One can further speculate from what has been said above that the function of a concept is to quickly sort experiences. In turn this implies firstly some classification of experience and secondly the fitting of new experiences into one of the classes. Finally, one can speculate that this mental referencing, the bringing to bear of previous experience on a new situation, is a continuous and automatic process so much so that it is only really noticed when something slightly incongruous occurs, such as when a small child addresses a strange, adult male as 'daddy' because the child has only experienced 'daddy' but not uncles, grandfathers or any other men.

In their power to organize data, there are different types of concepts. There are artificial concepts and there are natural concepts. Artificial concepts have well defined, criterial features. Thus there is little or no dubiety as to whether or not class membership obtains. Artificial concepts have been constructed by people in society to fulfil the technical, scientific, religious or legal needs of that society. Natural concepts, on the other hand, are characteristically ill defined in terms of defining features and clear boundaries. Natural concepts are everyday objects and informal events, instances of which depend heavily on function and context to determine class membership. Natural concepts are acquired informally and/or spontaneously while artificial concepts depend, to a much greater extent, on formal and/or structured teaching insofar as considerable verbal exposition may be needed to clarify relationships and/or make explicit the substratum of ideas (subordinate concepts) on which the concept rests.

The artificial/natural distinction draws attention to the fact that there are different levels of conceptualization. Those concepts which are derived from our sensory experiences of the environment are fairly low level. These are what Skemp (1971) describes as primary concepts. But these concepts which are derived from our intellectual abstractions from experiences in the outside world are of a higher order. These are what Skemp (1971) describes as secondary concepts. And herein lies a fundamental characteristic of mathematical knowledge. Its concepts are artificial and they are secondary. As Skemp (1971) points out:

> Much of our everyday knowledge is learnt directly from our environment, and the concepts involved are not very abstract. The particular problem (but also the power) of mathematics lies in its great abstractness and generality, achieved by successive generations of particularly

> intelligent individuals each of whom has been abstracting from, or generalizing, concepts of earlier generations. The present-day learner has to process not, raw data, but the processing systems of existing mathematics. This is not only an immeasurable advantage, in that an able student can acquire in years ideas which took centuries of past effort to develop: it also exposes the learner to a particular hazard. Mathematics cannot be learnt directly from the everyday environment, but only indirectly from other mathematicians. At best, this makes him largely dependent on his teachers (including all those who write mathematical textbooks); and at worst, it exposes him to the possibility of a lifelong fear and dislike of mathematics.

From what Skemp (1971) says it is clear that the mathematics teacher has a crucial role to play in helping the child to develop mathematical concepts. At one and the same time the teacher has to help the child form initial mathematical concepts from whatever everyday reality may exist for the learner and also to help the child relate previously assimilated higher order concepts to successively more complex concepts in the hierarchy. This abstract quality of mathematics is not lost on Resnick et al (1987) when they point out:

> There are not, strictly speaking, denotable objects in mathematics. For example, although one can point to a set of three things, and to the written numeral 3, those physical objects do not in themselves have the property of number. Number is a strictly <u>cognitive</u> activity. People construct this cognitive entity, the concept of number, without the benefit of any physical numbers to inspect or analyse. Yet number is the basic object of arithmetic. So we have in mathematics a domain in which, from the very beginning, people must reason about objects that exist only as mental abstractions.

The conceptualization of addition or subtraction, or for that matter, any other mathematical topic is not then going to be an abrupt metamorphosis from 'not having' to 'having' possession. In other words mathematical conceptualization for any given topic is not going to be an instantaneous once-and-for-all process which necessarily implies functional maturity. Since, according to Skemp (1971) all mathematical concepts are secondary concepts and since the formation of secondary concepts depends on being able to collect together lower order concepts which in turn have been detached from the sensory experiences from which they originated, it should not be surprising that mathematical conceptualization is a complex, lengthy business in which several years may lapse, according to Coon & Odom (1968), between "the emergence of a concept and its relative stability".

Some language to describe the mysterious mechanisms of conceptualization.

The term, conceptualization is respectable and useful as a short hand referent which alludes to learning and/or understanding in some comprehensive fashion. It is less than helpful as it stands, however, for describing unobservable cognitive functioning. To suggest that conceptualization has or has not taken place requires further language to talk about the extent to which the individual has made sense of his/her environment. A number of psychologists (notably Flavell, 1971; Flavell & Wellman, 1977; Skemp, 1971; Brown & DeLoache, 1983) have indeed endeavoured to set up such language. But, as could possibly be predicted, usage of terminology by leading proponents results in overlapping of meaning, synonymity of meaning and confusion of meaning which leaves the reader with feelings of frustration in understanding the nature of conceptualization. There follows then a consideration of some of the terms which can be found in the literature on conceptualization. There will firstly be some brief description of the terms **evocability, utilizability, intuitive intelligence, reflective intelligence** and **metacognition** with stipulative definitions where appropriate and secondly an attempt to integrate the meanings of these referents in terms of the significance of this research.

(i) Evocability: While the term evocability may rightly be referred to as a bit of jargon (with all the contemptuous connotations of the referent 'jargon') the verbs, evoke, evocate, evocable and their associated adjectives are to be found in the lay person's lexicon. The essential meaning of these words is to arouse, to summon up or to call forth some memory, emotion or answer from the past. There is also, in dictionary definitions, the suggestion that 'magical' or 'spiritual' powers are at work when the memory, emotion or answer is being recalled. Although no such mysticism is intended here in the use of the term evocability, its meaning is nevertheless close to the general notion of 'evoking'. The evocability of a concept refers to the individual's ability to trigger the concept into operation or, in perhaps simpler terms, to retrieve from long term memory pieces of knowledge which would or could help the individual to execute some task or other. For example, the car driver going on a long, previously untravelled journey might know that an ability to read maps would or could help in the plotting of the route. (The alternatives of would or could are used advisedly, for reasons which will hopefully become clearer later on). Another example might be that in order to have one's brand new electrical appliance function at all, the appliance needs to be fitted with a plug and so that one can connect the plug safely a knowledge of 'live', 'neutral' and 'earth' wires is needed. Perhaps another way of expressing what evocability means is in terms of knowledge of underlying principles.

Flavell (1971) considers that conceptualization can be described in terms of degree of evocability:

> Low evocability would mean that only a very small 'easy' subset of the entire range of tasks or problems soluble by that concept will as yet stimulate the child to retrieve the concept from long-term memory and attempt to use it as a solution procedure; high evocability would mean that the concept is now readily retrieved for possible use with respect to most relevant problems, even when the concept-to-problem fit is partly camouflaged by task or other variables.

In an arithmetical context then, the child who recognized that an addition operation was required when the numbers involved were integers less than ten but did not do so when they were of greater magnitude, could be described as being able to evoke the concept of addition at a low level only. Equally, the child who recognized that an addition operation was required when the numbers were into thousands or included fractions of numbers could be described as being able to evoke the concept of addition at a high level. The foregoing scenario suggests that levels of evocability can be equated with the magnitude of the numbers and/or the complexity of the arithmetic. Such a suggestion is undoubtedly a restrictive exemplification of what Flavell was saying. There may well be other 'dimensions' on which levels of evocability can be differentiated, but which in this instance have not been explored. Nevertheless the critical point to be made is that some evidence for the presence of a concept does not of necessity mean that the concept is functionally mature. Whatever 'dimensions' there may be on which to differentiate levels of evocability, it would be reasonable to assume, in the light of Flavell's distinction between high and low evocability, that young children, in the early stages of schooling, would be able to evoke mathematical concepts at a low level only.

According to Skemp (1971) there are two ways in which a concept can be evoked, or start functioning. The first is by experiencing an example of the concept in which case the the concept is evoked by classifying the particular example. For example, in order to describe similarities and differences between a Rolls-Royce and a Mini, the individual would have to recognize that both are example of the concept, car. Without the ability to activate the knowledge that cars are (usually) four wheeled vehicles which are powered by fuel, and which come in all shapes and sizes, and of which Minis and Rolls-Royces are but two examples, the task of making intelligent comment on Minis and Rolls-Royces would be impossible because the concept of car had never been activated. A second means of evoking a concept is, for Skemp (1971), by "hearing, reading or otherwise making conscious the name, or other symbol for the concept". This second means of evoking a concept is a specifically human phenomenon due, Skemp (1971) argues, to the human ability "to isolate concepts from any of the examples which give rise to them". Moreover, Skemp argues that this second means of activation is crucial in the process(es) of developing conceptualization because:

> Only by being detachable from the sensory experiences from which they originated can concepts be collected together as examples from which new concepts, of greater abstraction, can be formed.

(ii) Utilizability: Just as for evocability, the derivation of utilizability has its place in common parlance. The verb, utilize is regularly understood as 'making use of'. In terms of conceptualization, utilizability then refers to the ability to make effective use of the 'knowledge' required to solve a particular problem or carry out a specific task. The utilizability of a concept refers to the tactical application of the 'knowledge' rather than to the executive management of the 'knowledge' which is required in the evocability of a concept. The utilizability of addition and subtraction concepts would then be evidenced by the child knowing how to effect appropriate solutions to the operations of addition and subtraction when these operations were embedded in verbal contexts.

(iii) The relationship between evocability and utilizability: Flavell (1971) believes that the evocability of a concept and the utilizability of a concept can operate independently. He also, however, implies that there is a relation between evocability and utilizability when he states that utilizability is effected once the individual has "sensed the concept-to-problem fit". This slight confusion as to how independent/dependent evocability and utilizability are, perhaps hinges on the means by which the concept is evoked. Returning briefly to Skemp's (1971) descriptions of how a concept can be evoked, there is firstly the evocability of a concept by classifying a particular example one encounters. In this simpler sense it follows that utilizability can be effected only after the concept has been evoked. One cannot apply a concept if one cannot recognize any instances or examples of the class. Being unable to recognize any instances of a particular class can only mean that the class, or concept, is not in one's cognitive repertoire. To this extent, utilizability and evocability would appear to be closely related. However, for Skemp (1971) there is also the evocability of a concept by bringing into conscious awareness the referent for that concept. In this more complex sense it seems that one can entertain concepts without necessarily using them. For example, one might be fully aware that it is knowledge of electricity rather than some other knowledge which should be applied in fitting a plug to an electrical appliance, and yet be unable to utilize such knowledge because one has forgotten or never learned the electrical specifics involved. Equally, that one <u>knows</u> that map reading could help one in navigating a journey does not mean that one necessarily has the skill to use the map effectively. To this extent then, evocability and utilizability can be independent entities.

What seems to be emerging from this exploration, in simple language, is that for the individual to really utilize a concept (and not merely to rely on routinized, algorithmic skill alone) he/she must be able to evoke the concept. However, evocability can be conscious or unconscious. If the evocability is conscious the individual is likely to experience success in the task solution because he/she can state, express, articulate or be aware of what is required. If, however, the evocability is at an unconscious level the utilizability <u>may</u> be appropriate. But it may instead be inappropriate, chance alone determining the individual's task solution since the individual is not consciously aware of the principle(s) upon which the utilizability is structured.

(iv) The overemphasis on utilizability: Historically, in the field of education, those concerned with mathematical conceptualization (McIntosh, 1971; Brown, 1979; Riley et al, 1983) have been concerned exclusively with the utilizability of concepts. This is implicit:

in Vergnaud's (1982) view that "the ability to solve problems . . . is the best criterion of the acquisition of concepts";

in Dickson et al's (1984) view that "there has been, and still is, much emphasis placed on children becoming skilled in the standard written procedures of computation regardless of whether or not they understand the basis of such techniques";

in Desforges & Cockburn's (1987) view that children's "capacity to use their skills appropriately to recognise, represent and solve problems" is limited.

While of course, concern with the utilizability of concepts is desirable and even necessary, the emphasis, in the field of education, on utilizability to the exclusion of evocability may be hampering the child's growing conceptualizations in general and mathematical conceptualization in particular. Given Resnick et al's (1987) argument for number being a strictly cognitive entity and given Bryant's (1985) conclusion that performance in most cognitive tasks depends on both possessing the necessary skill(s) and recognizing when the skill is needed, it seems that evocability of a concept (that is, being consciously aware that this particular concept is appropriate in this particular instance) has been neglected both in the study of mathematical conceptualization and in the development of mathematical conceptualization in children.

Hitherto, the recognition of what is required on the part of the subject performing a cognitive task has been inferred by researchers. The presence or absence of a concept (in whole or in part) has been inferred from successful or unsuccessful overt performance on some behavioural index. To some extent this is perfectly correct since the field of cognitive psychology is exclusively concerned with mental functioning which of itself cannot be physically inspected. However, conclusions drawn from what abstractions are inferred to exist cannot really explain the evocability of concepts, if the distinction between evocability and utilizability is accepted. While such conclusions can account for the utilizability of concepts (which is essentially an exploration for/description of the behavioural manifestations of a concept's existence in the cognitive repertoire) evidence for the evocability of concepts - in Skemp's (1971) more complex interpretation of the process - must come from the subject's direct expression of his/her awareness of the particular concept's availability to him/her.

(v) Intuitive and Reflective Intelligence: Conceptualization in its fullest sense, according to Donaldson (1976) involves more than being able to respond to new and increasingly more complex groupings of stimuli. For Donaldson, conceptualization also involves a conscious awareness "not only of objects and events in the real world but also of our own thinking about these things". The notion of conceptualization embracing both representations of the world and an awareness of the representations themselves finds a parallel in Skemp's (1971) notions of intuitive and reflective intelligence. Intuitive intelligence is operating when "we are aware through our receptors of data from the external environment". Any task solution in which the individual is successful "without any awareness of the intervening mental processes involved" requires intuitive intelligence and to this extent intuitive intelligence would seem to resemble utilizability. Reflective intelligence, on the other hand, is operating when "these intervening mental activities become the object of introspective awareness", and would seem to resemble the complex, conscious means of evocability. These 'apparent' resemblances should not, however, be thought of in terms of simple substitutions or alternative referents. Rather the parallel comes from the 'whole' and not the 'parts'. Together evocability and utilizability would seem to 'add up' to the same as intuitive intelligence plus reflective intelligence. Conversely, the (qualitative) difference between intuitive intelligence and reflective intelligence would seem to be like the difference between utilizability and evocability.

According to Skemp (1971) this second order functioning of intelligence, that is, reflective intelligence is extremely powerful as a facilitator of conceptualization because, once one can at least to some extent reflect on one's own thinking one can:

firstly communicate one's concepts/schemata with another;
secondly build refinements onto existing concepts/schemata;
thirdly replace old concepts/schemata with new ones;
fourthly correct errors in existing concepts/schemata which will allow subsequent, improved task solutions.

Reflective intelligence would seem to involve a consideration of the <u>form</u> of task solution rather than the content, whilst intuitive intelligence seems tied to a consideration of the <u>content</u> of task solution rather than the form. In this distinction between intuitive and reflective intelligence we are again reminded of the relationship between the utilizability and evocability of concepts; utilizability being 'intuitive' and evocability being 'reflective'.

According to Skemp (1971), in any topic area, intuitive intelligence develops first and the development of reflective intelligence follows. However, the development of reflective intelligence is thought not to occur at all before adolescence. If this really is so then the young child is distinctly disadvantaged in that he/she cannot consciously communicate, refine, replace or update his/her thinking. The only progress that can be made has to be tied to content and context with the the resulting probability of success being a function of chance!

In terms of the emergence of intuitive and reflective intelligence, a rule of thumb would have to be that intuitive intelligence emerges first. After all one cannot reflect on concepts which are not yet established or formed in the mind of the individual. Nevertheless it is arguably an abdication of responsibility on the part of the professionals if they do not attempt to find ways of encouraging reflective intelligence at an earlier age than Skemp claims is possible. It is not enough to say, as Skemp does, that reflective intelligence is "relatively late in arriving" given the earlier list of 'mental advantages' with which the person in possession of reflective intelligence is equipped.

(vi) Metacognition: Being able to think about one's thinking or reflect on one's mental activities is now more commonly referred to as metacognition, a term coined by Flavell (1976). According to Flavell (1979), metacognitions are "not fundamentally different from other knowledge stored in long term memory" and they can be activated "as a result of a deliberate conscious memory search" or "unintentionally and automatically by retrieval cues in the task situation". More importantly, however, Flavell (1979) claims that metacognitive knowledge "may and probably often does influence the course of the cognitive enterprise".

It would seem from Flavell's (1971) notions of the utilizability and evocability of concepts, that metacognition does not have to involve utilizability, which is essentially the translation between a mental representation and an object or event in the real world, or as Skemp (1971) would put it, the use of intuitive intelligence. It would seem, however, that metacognition does involve evocability. Bringing into consciousness "the name or other symbol for the concept" (Skemp, 1971) either "as a result of deliberately conscious memory search" (Flavell, 1979) or "unintentionally and automatically" (Flavell, 1979) would seem to suggest that evocability is at least a part of metacognition. It hardly seems plausible that one can reflect on one's own mental processes without the essential tools of being able to detach concepts from the experiences which gave rise to them and being able to entertain such concepts in differing permutations.

Given Flavell's (1979) claim for the influence of metacognition on cognition, and given that evocability may well be a part of metacognition, it follows that the evocability of concepts, rather than just the utilizability of concepts should be given prominence in the study of conceptualization. The study of mathematical conceptualization is but one instance of this. Garofalo & Lester (1985) argue that "purely cognitive analyses of mathematical performance are inadequate because they overlook metacognitive actions". They maintain that the assumption that good tactical skill constitutes good mathematical performance is highly questionable; and believe that metacognition may account for a significant part of the "mental activity underlying the application of algorithms and heuristics".

Not only now are there claims for the influence of metacognition on cognition, there is the further claim for the positive influence of metacognition on mathematical conceptualization. If, as was argued earlier, evocability is a part of metacognition, mathematical conceptualization requires both to be studied and developed in children in terms of utilizability (in which terms it is already well documented) <u>and</u> in terms of evocability. Turner (1984) makes a similar point when she states that in the education of the child,

> what is required is a two-pronged attack, first by exposure to the forms of experience thought to give rise to the basic categories of human understanding, which are themselves a precondition for subsequent learning; and

secondly by alerting children to the nature of learning tasks in school.

(vii) Metacognition and the young child: Although the term, metacognition, is, even in psychological parlance, relatively new, the idea itself is not. As Donaldson (1976) points out, "the ability to reflect on one's own reflections is placed by Piaget at the heart of his account of mature adult thought". Metacognition would not, in Piagetian theory, be available to the individual before the stage of formal operations. For example, Piaget (1928) found that children of seven years of age could correctly solve a translation problem such as '3 boys are given 9 apples, how many will each have?' but could give no comprehensible, let alone coherent, account of how they had obtained their solutions. Subsequent research by others has demonstrated that primary school children are quite inept at monitoring their own cognitive performance. For example, Karabenick & Miller (1977) found that more than half of their 5-, 6- and 7-year old subjects were unaware that they did not understand the message given to them. Reid (1966) found that some children did not understand what their parents were doing when they held a newspaper in front of them. Renwick (1984) found that some children expected to be able to read after their very first day at school. Garofalo & Lester (1985) cite considerable evidence all of which suggests that it is only towards the end of primary education that the child begins to become aware of his/her mental activity. Such findings present a somewhat pessimistic picture of children's metacognition.

Brown & DeLoache (1983) however, are more optimistic. They discuss the phenomenon of metacognition in general and specific terms. Generally, there are basic metacognitive skills such as checking (did it work?), monitoring (how am I doing?) and reality testing (does this make sense?), which although "basic characteristics of efficient thought" are nevertheless "transituational". Because the child has to learn the various metacognitive skills themselves and additionally learn that these skills are almost "universally applicable", it should not be surprising that the young child is overloaded in terms of processing. The metacognitive problem for the young child is not so much an executive one as a tactical one. Brown & DeLoache (1983) liken the young child to a novice in whom the lack of "intelligent self-regulation" is, they claim, a defining characteristic.

In specific terms, Brown & DeLoache (1983) point out that tasks vary in the degree to which metacognitive control is required. They cite, for example, retrieval of objects from the environment as being easier than retrieval of information from memory. Brown (1978) and Brown & DeLoache (1983) do suggest, however, that it is possible to teach even young children some elementary metacognitive skills. Wolman et al (1975) found that young children knowing that they would later have to recall the locus of an object, performed better than those not so instructed. Cosgrove & Patterson (1977) found that if children are shown how to ask questions they become adept at doing so when they need further information.

Nisbet & Shucksmith (1984, 1986) are also concerned that metacognitive skills should be considered an integral part of the teaching/learning activities of teachers and children in school. They do not indicate at what age, for the child (!), this should start but they strongly suggest 'the sooner the better'.

Because the area of metacognition is not, at the moment, well defined (Brown, 1978; Flavell, 1979; Nisbet & Shucksmith, 1984, 1986; Garofalo & Lester, 1985), it is not at all clear quite how metacognition can be developed in and investigated in young children. However, given its suggested significance in conceptualization in general and in mathematical conceptualization in particular, and given the findings that even very young children can begin to learn to use metacognitive control, it would seem reasonable that a raw measure of metacognition could be gleaned from the individual's ability to evoke the concepts that are being/could be used in reflecting on a task.

The language for describing conceptualization seems, then, to fall into two distinct but not discrete groups. First there is the language which describes the **form** of conceptualization, language such as **evocability, reflective intelligence** and **metacognition.** Then there is the language which describes the **content** of conceptualization, language such as **utilizability, intuitive intelligence** and **cognition.**

Conceptualization

Metacognition	**Cognition**
Reflective Intelligence	**Intuitive Intelligence**
Evocability	**Utilizability**

Conclusions on Conceptualization

Grappling with a phenomenon which is not directly observable is no mean task. By inference and deduction one attempts to make claims for a competence which can only make itself evident through performance, and even then the performance may not truly reflect the competence. Such is the tentative status of conceptualization. And yet both of its strands, cognition and metacognition are the subjects of continuing investigation. The development of cognition has become a distinct and respectable area of psychology. The development of metacognition, hitherto referred to in somewhat oblique terms, is fast gaining credence. Our knowledge of both owes much to the work of Piaget.

While it is much debated that children between birth and adolescence progress through immutable stages of qualitatively different types of thinking, Piaget's central concepts of assimilation and accommodation nevertheless remain robust. The individual interprets events in his/her environment in terms of an existing frame of reference but if this is not altogether possible, the frame of reference itself can alter/change/adapt. These continual mechanisms of assimilation and accommodation constitute the expansion of the conceptual system, a system which has not only to construct complex representations of the world (cognition) but also has to take conscious control of such cognition and ultimately of itself (metacognition).

In the development of cognition, Piaget and his followers claimed that appropriate experience was vital since only the child's active engagement with the environment (rather than inculcation by others) would promote real conceptual growth. In the development of metacognition (though nowhere near so clearly delineated) Piaget suggests (Donaldson, 1976), and others have substantiated, that talking about one's physical and mental activity is critical for progress. Further detail of Piaget's contribution to our understanding of children's mathematical conceptualization will be the substance of the next chapter.

In summary:

(i) mathematical conceptualization is seen as extending over ontogenetic time and as being complex insofar as mathematical concepts are abstract;

(ii) mathematical conceptualization, as one instance of conceptualization per se, is not a directly observable process;

(iii) conceptualization can be described in terms of cognition and metacognition, the latter of which has not been delineated in any comprehensive way;

(iv) the development of metacognition is, it seems, important for mature conceptualization but the early seeds of metacognition can be evidenced in very young children;

(v) what is known about conceptualization is rooted in the work of Piaget.

THE CONSTRUCTIVIST VIEW:

classification, seriation and conservation

The complexity of conceptualization has been touched upon in the previous chapter. Conceptualization is a phenomenon which is of interest to those who are involved in education, and can be explored from a variety of different perspectives: from the perspective of cognitive psychology with its interests in memory, perception, mental representation, language and artificial intelligence; from the perspective of social psychology with its interests in group dynamics, attitudes, self-concept and social constructions of reality; from the perspective of developmental psychology with its interests in learning, individual differences and qualitative change between birth and adolescence. This chapter is concerned to consider the implications of conceptualization for school mathematics and hence will take a predominantly 'developmental' perspective.

Children, on entering the Scottish system of formal schooling at approximately 5 years of age, are typically introduced to a mathematics curriculum which embraces Shape, Measurement and Number. Shape and Measurement are assigned relatively minor weightings, while Number is assigned a position of primacy. To begin to understand something of the rationale underlying the early mathematics curriculum it seems reasonable to begin with the work of Piaget, since as Hughes (1986) points out:

> for some time now Jean Piaget has been regarded as one of the leading authorities on the question of how children learn mathematics.

For Piaget (1952), the child's understanding of number involves the simultaneous development of the ability to classify, the ability to seriate and the ability to conserve; and the superordinate ability to integrate these subordinate abilities to express relations.

At its simplest level, the ability to classify refers to being able to abstract a common criterion from a variety of criteria available and to sort entities into a set according to the criterion. For example, to be able to classify according to colour requires the ability to recognize different colours and sort accordingly, whilst at the same time disregarding other attributes which the entities to be classified may or may not share.

The ability to seriate is a refinement of the ability to classify. Seriation demands a recognition of the relationship between and among members of the class. For example, a group of people may vary in height and can be arranged in order from smallest to tallest.

The ability to conserve is the recognition that attributes such as number, weight, area and volume will remain invariant (that is, remain constant) in the face of perceptual change, provided no real change has been made to the attribute in question. For example, 10 men are 10 men whether they are sitting close together or standing far apart from each other.

In addition to possessing a competence in each of these three abilities, the child, in order to develop an understanding of number, must, according to Piaget, be able to unite these abilities. Thus the child needs to understand that a cardinal number represents a class of entities which:

1. have an inherent relationship - that is the counting of the entities will indicate the total number;
2. can both be broken down into sub classes, and combined with other classes to make a 'superordinate' class;
3. can be enumerated with consistent accuracy however they be arranged.

According to Piaget (1952), the child largely develops his understanding of number and other mathematical concepts by himself, "independently and spontaneously".

Hughes (1983) suggests that Piaget's claims are widely accepted and supported:

> the idea that mathematical concepts are acquired through the child's mental growth - and in particular through activities involving concrete objects - is taken as virtually axiomatic by most nursery and infant school teachers. The majority of infant school mathematics schemes start off with very concrete activities, such as matching objects on a one-to-one basis, or sorting them

> into sets. These activities are intended to develop the young child's general concept of number, as measured by a Piagetian conservation task. It is only when children seem to have grasped the idea of number conservation that they are considered ready to start on addition and subtraction.

Hughes seems to describe the Piagetian influence as being interpreted by teachers in a linear fashion: concrete activities allow mental growth which in turn allows concept acquisition. But it seems impossible to distinguish between 'mental growth' and 'concept acquisition' in terms of observable evidence. It further seems impossible to distinguish between 'mental growth' and 'achievement'. The child's achievement means what the child knows or can do now as a measure of progress from previous achievement. Thus, if a child knows more and can do more than previously, that is <u>increased</u> achievement. It is from such increases in achievement that we <u>infer</u> 'mental growth' and for many people, in casual usage especially within education, the notions of 'mental growth' and 'increased achievement' may be regarded as broadly equivalent.

The lack of specificity in Piagetian theory

Piaget's conception of the child's intellectual development was never so specific as to indicate the precise age at which a given milestone was reached. Nor indeed could it be, since his primary concern was to understand how human knowledge is constructed. Crudely put, Piaget wanted to know, in the words of Lovell (1979):

> if knowledge results from the accumulation of small bits of information or whether there must already be a mental structure or reference frame inside which some new piece of knowledge can be meaningful.

To pursue this question, Piaget studied the cognitive development of the child through investigating the construction of the child's basic conceptualizations, rather than the build up of particular skills or the acquisition of specific pieces of information. To the extent that cognitive development is concerned to describe the intellectual changes which take place between birth and adolescence and, further, to try to explain how and why these changes occur, its frames of reference are not age specific.

However, reference is frequently made as to the capabilities of the 'older' or 'younger' child: for example, "the youngest children were found to have no idea of class" (Turner, 1984); "young children generally fail and older children generally succeed with the traditional transitivity problems of the type administered by Piaget and by Smedslund" (Bryant, 1974); "young children might have very great difficulty with the invariance principle" (Bryant, 1974). Whilst it is accepted that chronological age and its correlation with any cognitive competence does not allow one to infer that age is an antecedent condition for increasing cognitive facility, it would nevertheless be more illuminating if one were to understand what is meant by the terms 'older' and 'younger' children. It is, further, unclear, what is meant by the term, 'capabilities'. Is this some undeveloped faculty which has the **potential** for manifestation? Or is it what the individual **can actually demonstrate**? In other words, we are back at the age-old competence/performance argument which in turn, has implications for the behavioural indices that are to be taken as evidence and from which inferences are to be made concerning unobservable mental activity.

The competence/performance can never be fully resolved because it is, in part, a function of differing predilections within the field of psychology: applied psychology is necessarily concerned with performance, while theoretical psychology is necessarily concerned with competence. Perhaps for the teacher who is wishing to make **use** of psychological findings there is a compromise. And that is, given that one cannot theorize about some unobservable mental functioning until one can comprehensively determine the specific situations in which performance reflects (albeit impurely and imperfectly) the underlying competence, there has to be a strong emphasis on valid diagnosis of the performance(s). Smedslund (1969) expresses this idea neatly:

> The relationship between any set of behavioural indices and a mental process is an uncertain one, and a diagnosis will always have the status of a working hypothesis.

The ambiguity of age in relation to milestones in conceptualization is partially addressed by Flavell (1977). His commentary on the Piagetian system, suggests that children in early childhood (from approximately 2 to 6 years of age) show "some striking cognitive immaturities" when compared to children in middle childhood (from approximately 7 to 11 years of age). In other words, one is left to infer that the qualitative changes which occur in the child's thinking from about 7 years of age are what account for the differences between the younger and older child's ability to classify, seriate and conserve. The younger child appears not to have these abilities whilst the older child does.

A further difficulty in understanding Piaget's precursors to mathematical understanding lies in the fact, as Hughes (1986) points out, that "Piaget never spelt out in detail how these ideas should be put into practice in the classroom". Hughes goes on to say that Piaget:

> has usually been interpreted as advocating a very late introduction to formal symbolism, with corresponding earlier emphasis on children's engaging in physical activities with materials such as sand, water, buttons, beads and bricks. It is assumed that pouring water from one container into another, or sorting objects into sets, will help develop children's mathematical concepts, and that they will proceed to formalisation only when they are conceptually ready.

This is not to say there have not been attempts to apply Piagetian ideas in education. There have (Schwebel & Raph, 1973; Kamii & DeVries, 1976), where the interest was not to have Piagetian schools in the way that there were and are 'progressive schools' or 'alternative schools', but to have teachers who were 'innovative' and 'imaginative' and who were not 'constrained' by schemes-of-work, educational objectives and performance indicators. To this day it is still part of the educational rhetoric that Piagetian ideas are of central importance in the primary school. But how justified is this assertion?

If it is the case that children must be able to classify, seriate and conserve before they can begin to develop an understanding of number and if it is the case that such precursors do not develop until approximately 7 years of age, then this author does not agree that belief in Piagetian theory is as widespread as exponents of the constructivist view would lead one to accept since, in Scotland at any rate, the typical Primary One child is expected to be able to add and subtract within 10, the typical Primary Two child to add and subtract within 20, and the typical Primary Three child to add and subtract within 100 (all with formal recording), by which time the typical Primary Three child may be barely 7 years of age! Perhaps regrettably, lip-service is paid to Piagetian ideas by using words such as 'readiness' or 'developmental'. These terms may suggest an adherence to Piagetian theory and therefore give a gloss or legitimacy to the curricular activities planned for children without there being any real attempt to take account of the subtle differences among children and the qualitative differences between children and adults. But why should this be? Are the ideas poorly understood by those of us in education? Or are the ideas themselves not really worth applying?

In an attempt to reach a considered opinion as to the importance of classification, seriation and conservation in the child's understanding of number, it is worth examining how these abilities are empirically demonstrated.

Classification

There can be little, if any, doubt that classification is a basic, organizational strategy in human thinking. Studies have shown that people have strong spontaneous tendencies to organize stimulus items into categories and subcategories (Gregg, 1975; Baddeley, 1976). Nor is this phenomenon only observable in experimentally induced situations. Morton & Byrne (1975) found that housewives, when asked to list the items required to equip a house, systematically grouped the items either according to categories such as furniture, linen, china etc or according to the place in which the items would be put such as bedroom, kitchen etc. Intuitively this appears eminently reasonable. If as humans we made no effort to categorize, then all new stimuli which were being attended to would have to be regarded as completely novel phenomena, totally unrelated to what had been experienced previously.

For Piaget, (Donaldson, 1978) the significance of classification for the understanding of number lay in the inference that if there are two or more sub-classes each of which contain at least one member, then the number of objects in the total class has to be greater than the number in any sub-class. It was Piaget's contention that children before the age of six or seven years could not compare a sub-class with a total class. The typical empirical test for this finding is to present the child with say 6 red flowers and 4 white flowers and then to ask the child, 'Are there more red flowers or more flowers?'. And typically, the young child answers that there are more red flowers because instead of comparing the sub-set of red flowers with the total set of flowers the child is comparing the subset of red flowers with the sub-set of white flowers.

From a naive and pragmatic viewpoint, there would seem to be no useful purpose served in asking anyone to compare a sub-group with a total group, while it seems much more reasonable to compare two sub-groups. One might, for example, want to know whether there were more red or more white flowers in order to decide which vase(s) to put the flowers in, or in order to decide where to site the flowers, or indeed in order to decide whether or not one should go and get more flowers of perhaps another colour.

Donaldson (1978) points out that the young child's typical response to the standard class inclusion task "has provoked a great deal of controversy, and much research beyond that which initially produced it". She goes on to cite McGarrigle et al (1978) who found that the manipulation of the wording of the class inclusion question in such a way as to vary the emphasis placed on the total class, produced significantly different responses. Their conclusion is that the child's interpretation of the task is of paramount importance in determining their performance. In other words, a far higher percentage of children (below the age of six) than had hitherto been evidenced could successfully perform on a class inclusion task provided the context was a meaningful one for them. That the salience of the context will influence if not wholly determine the category into which one will classify an entity is a point made by Anderson & Ortony (1975) who state that "there are so many ways in which every object can be classified. . . there are cases in which only the context will help us to determine how to classify an object".

The importance of classification then can be briefly summarized. The organizational strategy of classifying entities into categories is a central human activity. However, it is context dependent and failure on the standard class inclusion task does not mean that one cannot distinguish between subordinate and superordinate categories. Meaningful modifications of the task have demonstrated that children as young as three years of age can make the distinction (McGarrigle et al, 1978).

Seriation

The serial ordering, or seriation, of members of a class requires that the members be put into a sequence according to the property in question. For example different tins of beans could be sequenced according to weight (lightest -> heaviest), different strips of material could be sequenced according to length (shortest -> longest), different containers could be sequenced according to the volume of water or capacity of sand that they hold (holds least -> holds most). The significance of this ability is that it requires the recognition that Mary can, for example, be <u>both</u> shorter than Julie and taller than Susan: that is, that there is an ordinal relationship between the members of a set which allows one to co-ordinate separate judgements inferentially. Thus, in answer to the question, 'Who is the tallest girl?' it is possible to infer, deductively, from the above that it is Julie.

As Bryant (1974) points out, it is rather important in educational terms that children should be able to infer transitively since the children who cannot, "clearly cannot understand even the most basic principles involved in measuring things". He goes on to say that:

> there will be little point, for example, in teaching such a child how to use a ruler, because he will have no conception that different things could be compared with each other through their common relations to it.

The typical, Piagetian, transitive inference task is to compare two quantities (either different sizes or different weights) directly, A with B. Then one of these quantities, B, is directly compared to a third, B with C. Finally the child is asked about the relations between the two quantities which have not been directly compared, A with C. This last part is the point at which the child has to make the inference since in order to make a judgement about the AC relation the child must combine the information from the separate, direct comparisons between A and B and between B and C.

Bryant (1974) points out that, typically, "children below the age of approximately seven or eight years of age were not able to answer the inferential AC question". And he also stated that amongst many other psychologists and across several different experiments there is the "same developmental trend from consistent failure to consistent success".

Bryant (1974), however, argues powerfully that young children's failure in the transitive inference task is principally due to their poorly developed memory systems and that their success in the task is possibly due in some measure to chance. In a modified version of the standard task, Bryant had the child make four initial direct comparisons, A and B, B and C, C and D, D and E; gave the child a lot of experience with these comparisons and made the BD relation the crucial test of the child's ability to make inferences. As a result, Bryant found that children as young as four were able to make transitive inferences.

Nor does Bryant see any reason for thinking that children as young as four years of age "could not also make similar inferences about number". That the child does not cope with the ordinal relationships (which is essentially what being able to make a transitive inference involves) within number - that is understanding the necessity of the fact that if 4+2=6, then 6-4 must equal 2 - is, according to Bryant, due to memory failure rather than a competence lack.

As in the case of classification then, it would appear that children can develop the ability to seriate (albeit with support) before they come to school.

Conservation

For Piaget (Turner, 1984), conservation was the "centre of rationality": and according to Pinard (quoted in Turner) "extends beyond the few privileged domains to which it is customarily restricted". In the typical conservation experiment (in this instance to test the conservation of volume), the child is shown two identical containers, A and B, filled with identical quantities of liquid. The child is asked to confirm that both containers hold the same amount of liquid, and then watches while the liquid in container B is transferred to a third container, C, which is taller and thinner. On being asked which container now holds more, the young child tends to judge the quantities in A and C to be unequal and most often states that C, which has the highest level, contains more. For Piaget this failure in the conservation task is largely attributed to the child's inability to grasp the logical principle of invariance; that quantities remain unchanged over perceptual transformations. Piaget's interpretation has been variously challenged:

1. Bruner, Olver & Greenfield (1966) do not believe that failure in the conservation task is due to a lack of understanding of the principle of invariance. According to them, from the different modes of mental representation (enactive, iconic and symbolic) available to people, the

enactive mode (which is based on internalized actions) and the iconic mode (which is based on internalized perceptions or images) predominate in the cognition of the younger child. When erroneous judgements in conservation tasks are made they are due to attention being focussed on the perceptual aspects of the task. The symbolic system 'knows' that the liquid is the same after being poured into the new container but the iconic system insists that it is different and the iconic system dominates the response. Bruner et al demonstrated that if the perceptual differences were concealed from the child by a screen which allowed him/her to see the pouring take place, but not the resulting discrepant levels, he was more likely to judge correctly that the quantity of the liquid poured from one container to another remained the same. Thus it is Bruner et al's contention that correct conservation responses emerge only if the perceptual evidence is weakened, or later when the verbal system (symbolic mode) becomes stronger.

2. Bryant (1974), from extensive study of the invariance principle, argues that young children seem to have two rules and use them in different situations. One rule is that if the child sees only <u>one</u> quantity (instead of two as in the traditional conservation task) which is then transformed, he/she applies the principle of invariance -

that is the child knows and asserts that since nothing has been added or taken away then nothing has changed. The other rule seems to be that if the child is required to compare <u>two</u> quantities he/she uses a perception-type cue (of 'looks' bigger, taller, longer, etc.) Bryant maintains that the young child does not realize that these rules are inconsistent and so applies the first rule when he/she sees a quantity transformed and the second when he/she has to compare two quantities.

3. Donaldson (1978), Donaldson & Balfour (1968) and Donaldson & Wales (1970) see linguistic confusion as being linked to nonconservation. Their research indicates that relative terms such as 'more' and 'less' do not have the same connotations for young children, as they do for adults. Clearly, then, if the child literally does not understand what the words 'more', 'less' and 'same' mean then he/she will not be able to carry out the task. Additionally, however, Donaldson and her colleagues argue that the conservation task (and for that matter the class inclusion task also) requires 'disembedded' thought - that is the child must think about the language used by the adult independently from the context in which it is being used. Prising the thought from its context such that the thinking becomes the "manipulation of meaningless symbols" is, as Donaldson (1978) points out, difficult even for

adults and yet, in our society, it is the type of thinking which is highly prized in our educational system. The ability to reason syllogistically is an obvious (albeit extreme) example.

The importance of a salient context in which to demonstrate conservation is well exemplified in two studies. In Cohen's (1967) work a tea table experiment was carried out in which children of 4½ years of age had to share out quantities fairly. They were extremely accurate in compensating for the differences in the shape of the containers provided. The correct nonverbal conservation responses were in contrast to the typical erroneous verbal responses given by a matched group of children who performed on the standard Piagetian version of the task. Similarly in Donaldson's (1978) work, with the transformation of the stimulus material carried out by 'Naughty Teddy' so as to suggest to children that the transformation had been accidental, the number of children successfully demonstrating the principle of invariance was markedly greater than in the traditional version.

These two pieces of work illustrate the point alluded to at the beginning of this section - that the principle of invariance has application in everyday life. It is not so much a rule of logic which, once understood, will serve in any situation but, more, a convention which will have application in some contexts but not in others. For example a heap of lego bricks is a tower when the child has built it up but not when he/she has knocked it down again. Similarly, each individual when described by his/her given name retains his/her identity in spite of changes in size and appearance through life, but when reference is made to the person's physical appearance as it changes from childhood through to middle age, then invariance is not preserved. The notion of invariance is much 'bigger' than what can be encapsulated in a standard Piagetian conservation task. How the child performs on such a task will be influenced at least by the language used, the child's perceptions of the experimenter's intentions and the interaction between the two.

In summary then, it would appear that while classification, seriation and conservation are in themselves robust concepts, their elicitation is context bound. Perhaps what should be remembered is that the types of experiments which have classically demonstrated classification, seriation and conservation are tasks in a social setting and therefore both the behaviour of the experimenter and, more importantly, the child's interpretation of the experimenter's behaviour must be taken into account. The traditional experiments assume that the task which the experimenter is setting is the same task that the child is doing. Subsequent studies (as cited above) cast doubt on that assumption.

The relevance of Piagetian theory

The finding that children, even before school age can classify, seriate and conserve, ultimately leads one to question (in part at least) the relevance of Piagetian theory to early mathematical education. Certainly, Piagetian theory seems "to set limits on the kind of reasoning and understanding we can expect from children at any particular point in their development" (Resnick & Ford, 1984). This is all well and good as a general teaching principle, but it is altogether much too vague to allow one to draw from it teaching implications for specific mathematical tasks. And even those educationalists concerned to apply Piaget in the classroom are not sufficiently specific to enable practising teachers to see/understand what they should be doing in the name of Piagetian application. Kamii & DeVries (1976), for example, advocate a variety of musical, group and board games; describe some "situations particularly conducive to the construction of elementary number"; and expound half-a-dozen "principles of teaching"!

Hughes (1986), however, draws attention to the nub of the matter. He points out not that Piaget's theory is incorrect "but that it lacks immediate relevance for those attempting to cope with children's real difficulties in learning mathematics". Similarly, Groen & Kieran (1983) claim that there is a most serious gap "between Piaget's tasks and the tasks of school mathematics". They argue that although there may be a connection between Piagetian tasks and school mathematics, "it is not an explicit one". The Piagetian tasks "lack the face validity or direct correspondence possessed by a task such as addition or solving equations" (Groen & Kieran, 1983).

Groen & Kieran (1983) maintain that because Piagetian theory "provides no apparatus for bridging the gap" between empirical findings and school mathematics, and because Piagetian theory posits the notion that intelligence develops through a sequence of stages and substages, it is perfectly possible to <u>infer</u> a bridge between the two. They list this inference in terms of the following assumptions:

1. Piaget's "intelligence" is the same as that required by school mathematics.
2. Intelligence develops according to Piaget's "main sequence" of stages - sensorimotor, pre-operational, concrete and formal operations.
3. The stages of this main sequence, together with their substages, define a particularly ordered set of slots into which the tasks of school mathematics can be inserted. This defines the level of intelligence necessary for successful performance on any given task.
4. A given level is attainable only by going through all prior stages and substages.
5. Performance in school mathematics can be improved by

explicitly teaching appropriate Piagetian tasks as generated by assumption 3 - for example, improving addition by teaching conservation of number.

Having, by inference, 'forced' a connection between Piaget's theory of how knowledge is structured and his theory of the sequence of any set of developmental events, it then becomes (relatively) easy to understand why assumptions 3, 4 and 5 have become almost central tenets in what now can only be referred to as quasi-Piagetian approaches to school mathematics. This is not to say that Piagetian tasks might not be essential components of tasks in school mathematics. They may, somehow, very well define necessary conditions for success but it is becoming quite clear that traditional Piagetian tasks are not easily mapped onto conventional school mathematics.

The appreciation that Piagetian measures of conceptualization in children do not have immediate application in the classroom is, however, a relatively recent phenomenon. Before Hughes (1986) and Groen & Kieran (1983), loyalty to the Piagetian framework would seem to have resulted in an almost unjustifiably reverent attribution of recommendations for good teaching practice to the findings of Piaget. For example, Lovell (1972) believes that the Piagetian, cognitive-developmental model does have something to say to teachers:

<u>firstly</u> he suggests a move from the formal classroom,

whole-class teaching to small group and/or individual teaching;

secondly he suggests that there be "opportunity for pupils to act on physical materials" - these being considered necessary for the abstraction of concepts;

thirdly he suggests that classroom conditions allow teacher-child and child-child interactions - these being seen as important in helping the child to organize his/her thinking, and in eliciting "the strategies of thinking" available to the child;

fourthly he suggests that "the initiative and the direction of the work must be the teacher's responsibility";

fifthly and finally he suggests that "alongside the abstraction of the mathematical idea from the physical situation, there must be the introduction of the relevant symbolization".

Perusal of Lovell's (1972) implications may imbue the reader with feelings of deja-vu, and for good reason. As McIntosh (1977) points out, recommendations of this sort were being made "sixty to one hundred years ago" so however laudable they may be they cannot be said to derive exclusively from Piagetian theory. At best they are not inconsistent with it and what they really serve to tell us is that while Piaget's contribution to our understanding of children's conceptualization has been enormous, any theory - including Piaget's theory - is merely a tentative statement that needs constant modification on the bases of evidence gained from the testing of its offspring hypotheses. Once a theory ceases to perform its function of incorporating new discoveries about human behaviour it can be set aside.

Since Piaget's assessment of the cognitive abilities of young children was on the basis of tasks which have no obvious or direct application in the teaching of mathematics, there seems little point, in the context of this study, in pursuing his theory any further. Suffice to say that while what can be implied from his theory in respect of mathematical learning, as spelled out by Lovell (1972), is encouraging, it is not radically illuminating. It therefore now seems appropriate to consider the findings of those who have studied the mathematical achievements which young children have directly demonstrated.

In summary:

(i) the constructivist view of number conceptualization states that sound understanding of number involves the abilities of classification, seriation and conservation;

(ii) research suggests that the ability to perform successfully in these tasks is a function of context, interpretation and memory;

(iii) the links between the concepts of class inclusion, transitive inference and invariance and the abilities to perform school mathematics may be tenuous, and are certainly not obvious;

(iv) without in any way underestimating the importance of Piagetian theory in the development of mathematical conceptualization, it is fair to say that Piaget's findings are of little immediate relevance to the teacher of mathematics.

THE NEO-PIAGETIAN VIEW:

counting

The purpose of this chapter is to consider the conceptual underpinnings of addition and subtraction from a perspective which is slightly different from that of Piaget. This perspective is **not opposed** to the Piagetian view: rather it has grown out of the Piagetian view and as such can be referred to as neo-Piagetian.

Gelman & Gallistel (1978), who have set about very thoroughly trying to establish what young children can do (as distinct from establishing what they cannot do), argue for the primacy of counting as the means whereby children begin to develop understanding of number. Moreover, they contend that children's early mathematical abilities develop in much the same way as we now understand early language development: by a system of self-generated rules.

Starkey & Gelman (1982) react strongly against the Piagetian view that counting processes are "rote processes, the products of which have no numerical meaning or utility to the child" (Starkey & Gelman, 1982). Indeed Gelman & Gallistel (1978) go so far as to say that children cannot reason about number, that is think in the abstract, algebraic sense, until they have developed an understanding of how numerosities are obtained.

This understanding develops through counting which, in turn, is governed by five counting principles. For Gelman & Gallistel (1978), the developmental process in understanding number is one of:

(a) learning the system of counting names;

(b) perfecting the use of counting principles that constitute counting;

(c) reasoning about number.

Each of these facets will now be explicated in turn, but in so doing it must be stressed that there is no implication that in actuality each follows the other in temporal sequence; in the sense of one being complete before another commences.

(a) Learning the System of Counting Names

According to Fuson & Hall (1983) number words have a variety of meanings and uses. The meaning of number words is determined by their uses in particular contexts. Fuson & Hall (1983) suggest the following contexts as ones in which young children acquire number words.

(i) Sequence Context - The English language words, one, two, three, four, five, etc in their conventional sequence, are learned. In this context, sequence words are relatively meaningless in that there is no correspondence between the words and the entities to be counted. Nevertheless sequence production activities are found to have a variety of (albeit low level) uses by children: spontaneously reciting a sequence to demonstrate to others the reciter's 'skill'; reciting a sequence to a pre-determined number in a game of Hide-and-Seek; reciting sequences in number rhymes - 1, 2, 3, 4 Mary at the Cottage Door, or 1, 2 Buckle my Shoe; reciting the sequence forward to find the number after 6, and backwards to find the number before 8. And even adults can be seen counting to 10 to control their temper! Fuson & Hall maintain that:

> each sequence production should contribute to the acquisition of the sequence and to its eventual fluent production but probably does not contribute substantially to any further knowledge regarding the number words.

(ii) Counting Context - The sequence number words are successively assigned to countable items. In other words there is a one-to-one correspondence between number words and entities to be counted. In young children, counting can be accompanied by pointing which according to Fuson & Hall, connects the entity existing in space to the word existing in time; or counting may be accompanied by the physical act of moving the entities from the pile of uncounted to the pile of counted entities. Over ontogenetic time, the dependence on physically marking the items being counted, fades, if it was ever there at all, since some even very young children count without demonstrating overt indications of counting.

(iii) <u>Cardinal Context</u> - The number word describes the total numerosity of the set of countable entities. Fuson & Hall cite evidence to show that children as young as two in western culture can use number words for small arrays in a cardinal context - for example, two shoes, two hands. However, as Fuson & Hall also point out, "the capacity to process numerosity information for small arrays does not mean that the child is aware that numerosity is a property of all sets". In other words the child may need help in generalizing from the different cardinal contexts he has experienced to abstract the concept of numerosity words as having context free meaning.

(iv) <u>Measure Context</u> - The number word describes the numerosity of the units into which some continuous dimension of an entity has been divided - for example, two cups of flour, four footsteps, three litres of milk, five gallons of petrol. The measure context involves firstly an appreciation of non standard units and later an appreciation of standard scales. In using non standard units such as cupfuls, say, to measure a quantity of flour, the child must realize that each and every cupful be filled to the top with flour. Thereafter all the cups of flour can be counted or, if only one cup was available in the first place, a tally kept of the cupfuls so far counted. In using standard units, particular aspects of the use of each scale must be learned - for example, one's waist does not measure 26 inches if the tape has not been positioned correctly all the way round the waist; one's body temperature is not constant in the face of differing environmental influences such as having a hot drink before a thermometer reading is taken and so on. Fuson & Hall argue that the measure context is so wide ranging in terms of the procedural and declarative knowledge required that:

> it seems likely that the measure concepts of children consist of a scattering of relatively isolated fragments, with little overall generality.

(v) Ordinal Context - The number words, first, third, ninth etc describe the relative magnitude or the relative position of a discrete entity within a well-defined, totally ordered set of entities in which the ordering relation has a specified initial point as in the fist man to walk on the moon, the second police car to arrive at the scene of the accident, the third child in the line, the fourth child in the family and so on. Fuson & Hall maintain that ordinal words beyond first, second and third are probably learned by derivation from the standard word sequence rather than being learned in sequence as cardinal words are.

(vi) Non Numerical Context - The number word is used as an identification. Post codes, telephone numbers, credit card numbers room numbers are but a few of the ubiquitous applications.

Given the variety of contexts in which the same number words are used, how then does the young child learn the system of counting names? According to Fuson & Hall:

> the child first learns the number word as several different context dependent words. Later these different meanings of the word become inter-related, resulting in a mature, closely connected set of meanings for that word.

(b) Perfecting the use of the Counting Principles

For Gelman & Gallistel (1978) it is the availability of certain principles as well as the ability to use the principles in concert which underpin the ability to count.

The Principles

(i) **one-one principle**

(ii) **stable-order principle**

(iii) **cardinal principle**

These first three principles are referred to as the how-to-count principles because they are procedural in that they specify the way to execute a count.

(iv) **abstraction principle**

This fourth principle is referred to as the what-to-count principle because it deals with the definition of what is countable.

(v) **order-irrelevance principle**

This fifth and final principle combines features of the other four principles.

The how-to-count principles

Being able to count involves having an understanding of one-to-one correspondence in assigning a distinct counting word or tag to each of the entities in the countable array. There must be one and only one tag used for each item in the array. In the sophisticated counter in western culture this means using the traditional counting words in sequence. However the one-one principle is not dependent on the traditional naming sequence. Gelman & Gallistel found that very young children used their own idiosyncratic sequences such as counting a two-item array by saying "two, six" and a three-item array by saying "two, six, ten". What Gelman & Gallistel noticed was that children using idiosyncratic lists used them in a systematic fashion - that is, "the same sequence of tags occurred trial after trial". This consistency in assigning tags across a count is what is meant by the stable-order principle. Moreover, Gelman & Gallistel found that children who used idiosyncratic lists "were better able to apply the stable-order principle than were children who used conventional lists of number words": the explanation being that "the child remembers a list of his own making better than one imposed from outside". Nevertheless, in the interests of arithmetical communication it is essential that the child does learn to use the conventional sequence of number words. Gelman & Gallistel acknowledge that the child "probably will require considerable practice before he is skilled in his use of the conventional list". The implication here is not that the child must be taught to count - he/she is already guided by the stable-order principle in counting - but that the child has to learn the conventions of counting. This will involve to some extent the rote learning of the sequence names although here again Gelman & Gallistel argue that after learning the sequence of the first twelve or thirteen conventional number words, subsequent number words are produced by generative rules. Fuson & Hall (1983) similarly mention that children seem to learn the one to nine pattern within the decade first and later produce the decades in conventional sequence.

Being able to assign tags in a one to one fashion and being able to do so in a fixed order is not all that is involved in knowing how to count. A crucial component is the knowledge that numerosity is a property of all countable entities. For the child to indicate the numerosity of the set he/she must be able to state that the final tag applied to the last countable entity in the set represents the set's total numerosity. This is what Gelman & Gallistel mean by the cardinal principle which can be understood as having become established from any of the following behaviours:

(i) being able to respond immediately with the correct cardinal or to a 'How many?' question about a set;

(ii) emphasis on the last word produced in counting by louder and/or slower pronunciation;

(iii) repetition of the last word in counting;

(iv) stating, without counting, the correct cardinal word after that same set has been counted on an earlier trial.

Fuson & Hall (1983) point out that the actual response to a 'How many?' question has two successive stages (at least for sets too large to subitize), the first being enumeration and the second being the reporting of the final tag as a cardinal word. They go on to say that the child who recounts to a repeated 'How many?' question may be viewed as having mastered only the first of these two stages. According to the Gelman & Gallistel model this would be indicative of the one-one principle and the stable-order principle only having been acquired by the child. And indeed, Gelman & Gallistel do state that "the cardinal principle, which presupposes the other two, should develop later".

Gelman & Gallistel believe that the how-to-count principles constitute a schema in the Piagetian sense: that children are intrinsically motivated to develop their counting abilities. The observed counting behaviour of children has shown that children count spontaneously in what to adults would seem purposeless counting activities, that they self correct, that they eventually learn to count accurately whether or not there has been planned and formal input to that end. Gelman &

Gallistel draw some parallels between the Piagetian model and their own. For example, Piaget's notion of children perfecting and practising newly developed schemata is seen as similar to spontaneous counting; Piaget's notion of the child's interpreting his/her world according to his/her existing frames of reference is seen as similar to the development of idiosyncratic counting lists; Piaget's notion of the environment impinging on the child and forcing change or 'accommodation' is seen as similar to the child being forced to adopt the conventional counting list.

This inherent need by young children to practise, consolidate, extend and apply their how-to-count schema is considered to be highly beneficial to the child and would seem to support the view that, historically we have underestimated children's mathematical abilities by failing to realize that the acquisition of early number concepts is, like the acquisition of early language, a process in which the child takes the initiative.

The what-to-count principle

The abstraction principle is the understanding that the how-to-count principles can be applied to any array or collection of entities, be they physical or non physical, heterogeneous or homogeneous. Gelman & Gallistel's interest in the abstraction principle stems from the fact that while for adults it is self evident that almost anything can be counted, some very eminent developmental theorists have argued that what the child sees as a collection to be counted is tied to his ability to classify objects and events into organized, criterial groupings. Typically it has been postulated (Bruner et al, 1966; Piaget, 1952) that children first classify according to the salient perceptual properties of objects, (such as colour and shape) and only later apply 'abstract' criteria (such as function, contextual association, logic etc) in their classification. Because Piaget, perhaps the chief protagonist, has emphasized the significance of classification in the development of number understanding, there has according to Gelman & Gallistel been "fostered the belief that children place restrictions on what can be counted". They do not argue that children do not place restrictions on what can be counted but they do argue that children do not restrict themselves to counting collections of identical objects. Gelman & Gallistel maintain that a complex classificatory schema need not necessarily mediate the ability to classify entities as 'things'. They state;

> It is possible to view the ability to classify the world into things and nonthings as a derivative of the ability to separate figures from grounds. In this case, the categorization of things as opposed to nonthings may well be among the earliest (most primitive?) mental classifications. A differentiated and ordered hierarchy of subcategories of things may well be a later development.

What Gelman & Gallistel seem to be suggesting is that the earliest form of classification is the ability to get from the environment any information such that some discrimination can be made but that this discrimination may well not be at the relatively sophisticated level of shape, colour, function, association and so on. Thus, for Gelman & Gallistel, classification as a precursor to counting is at a very much cruder level than was suggested by Piaget.

The order-irrelevance principle

The order-irrelevance principle refers to the fact that the order of enumeration is irrelevant. In other words it does not matter how you count the collection of countables so long as you count them all and count each once only. Appreciating that the order of enumeration is irrelevant shows an awareness that firstly the number names, one, two, three and so on are not inherent properties of the countable items but merely arbitrary and temporary designations; and secondly the total cardinality of a set is not affected by the order in which the countables are processed.

As well as demonstrating the ability to co-ordinate the how-to-count and what-to-count principles, the order-irrelevance principle demonstrates, according to Gelman & Gallistel, "an understanding of the fact that much about counting is arbitrary". In their empirical work Gelman & Gallistel found that children of 4 and 5 years of age were "remarkably good" at modifying the order of enumeration when the experimenter designated a specific object (say 2nd or 3rd in a linear array) as 1. Moreover such children could "invoke the principle in their verbal account". In other words they would justify to the experimenter that the number word assigned to each object was dependent on the specific order of enumeration in any given count. This is not to say that the execution of the count was always perfect, but then neither is it with adults!

It is Gelman & Gallistel's contention that in grasping the order-irrelevance principle the child knows what he/she is doing when counting. Furthermore Gelman & Gallistel believe that children possess this information by the time they come to school.

(c) Reasoning about Number

For Gelman & Gallistel, the child's ability to obtain representations of numerosity (which the child does by counting as outlined in parts (a) and (b) above) is a necessary prerequisite to his/her ability to reason arithmetically. Having grasped the how-to-count principles, the child routinizes his/her skill by increasingly applying the cardinal rule (that is counting aloud anything and everything in sight!)

and later by increasingly stating the cardinal number name without having counted aloud. Gelman & Gallistel stress, however, that the counting process is not an intrinsic part of reasoning but that counting provides the representations of reality upon which the reasoning principles operate. They delineate the numerical reasoning principles as follows:

(i) numerical relations;

(ii) operations;

(iii) solvability.

(i) Numerical Relations refers to the child's ability to draw comparisons between two numerosities. Gelman & Gallistel believe that young children can accurately compare two (small) numerosities and indicate whether or not the arrays are numerically equivalent. This occurs even if one of the arrays is made to 'look' larger as in having a 'long' line of 3 counters to compare with a 'short' line of 3 counters. Gelman & Gallistel argue that the child's judgement of the equivalence/non equivalence in numerosity is mediated by representations of numerosity, which are in turn derived from counting. If the child does in fact count the members of each of the two arrays rather than construct a correspondence between the members of each set (which is the Piagetian view) then the concept of conservation as it applies to number seems somewhat redundant. Moreover, Gelman & Gallistel maintain that if a non equivalent relation holds between the two numerosities, the child seems to recognize that an ordering relation holds. That is the child knows that one set holds more and another less - although the child rarely uses the nomenclature - by indicating the direction of inequivalence. That young children can, further, "use an ordering relation in an inferential manner" (Gelman & Gallistel) calls into question, once more, the significance attached to the results of the standard, Piagetian seriation task.

(ii) Operations refers to the child's ability to manipulate numerosities and to know whether the manipulation has affected the numerosities. For example the child knows that spatial rearrangements, colour changes and item substitutions do not affect numerosity. This is what Gelman & Gallistel refer to as the operation of identity. Similarly when confronted with a transformation that has altered the numerosity the child knows that either something must have been added (in the case of an increase in numerosity) or that something must have been taken away (in the case of a decrease in numerosity). In every case the child makes his/her decision by counting.

(iii) Solvability refers to the child's ability to repair the effects of the addition and subtraction operations. If the numerosity has been transformed by addition the child knows how to reverse its effects - by subtraction. Conversely if the numerosity has been transformed by subtraction the child knows how to reverse its effects - by addition. Again however, the execution may be imperfect. The child knows how to solve the problem (that is by addition or subtraction) but he/she need not necessarily know exactly the number needed to make the reparation.

The Piagetian and Neo-Piagetian Views

Clearly there are differences between the Piagetian and neo-Piagetian views on how children develop an understanding of number. For proponents of the Piagetian view, children do not understand, can not understand what addition and subtraction mean until they can conserve, seriate and make inferences about number. And this seems to happen at about seven years of age. What is more, the Piagetian view claims that any 'counting' which a child does before then is meaningless and mechanistic. By way of contrast the neo-Piagetian view emphasizes the central importance of counting, believing it to be based on a number of principles all of which most children will have grasped, at least in part, by the time they go to school. Not only, for the neo-Piagetians, does counting start at an early age, it is the basis of understanding:

> that non numerical transformations do not affect numerosity;
>
> whether an addition or subtraction operation is required;
>
> that addition and subtraction are complementary functions.

The pivotal role of counting amongst pre-schoolers and early schoolers has, further, been evidenced by Ginsburg, 1977; Carpenter & Moser, 1982; Case, 1982; Fuson, 1982; Ginsburg, 1982; Steffe et al, 1982; and Resnick, 1983. It is counting which allows young children to find solutions to various practical, addition and subtraction arithmetical problems.

Within this realm of counting there are a number of strategies which children will use spontaneously: in addition there is 'counting all' and 'counting on' (Fuson, 1982); in subtraction there is 'separating from/counting down from', 'adding on/ counting up from' and 'matching' to count the unmatched objects (Carpenter & Moser, 1982).

Much of the counting behaviour observed by neo-Piagetians has included finger counting. Finger counting, though not always a reliable procedure, seems to be meaningful to young children (Hughes, 1986). Ginsburg (1977) maintains that young children are "likely to try to count on their fingers" and that teachers should facilitate this in their young pupils. Hughes (1986) specifically states that teachers should show children "how to use their fingers more effectively" and that the "different methods of different children" should be made "the focus of class discussion".

While the implications of the Piagetian view are difficult to understand in terms of classroom application, the implications of the neo-Piagetian view are much more direct. The child comes to school with **some knowledge of and about** counting. In this, then, there appears to be a form or structure on which teachers of young children can build. Obvious questions for the teacher might be:

to what extent can each and every child in the class count?

what is the range of counting skill amongst the class members?

If, as the neo-Piagetian view argues, the ability to add and subtract has its origins in counting it is clearly important to investigate what children's counting performance is. The next chapter is given over to describing the present author's attempts to investigate the counting skills of the children in her primary one class.

In summary:

(i) the neo-Piagetian view sees counting as the basis of arithmetical understanding;

(ii) counting begins to emerge in children as young as 2-, 3-years of age;

(iii) counting skill is culturally transmitted - it is not the exclusive preserve of formal teaching - and is thought to emerge in children in much the same way as language develops; (iv) the child's ability to count should be exploited when he/she begins to participate in formal schooling.

THE BEGINNINGS OF THE EMPIRICAL WORK:

THE INITIAL STUDY

If not in agreement in all respects, the literature cited in the previous chapters suggests that mathematics is ubiquitous. It has power and it has precision. Furthermore, mathematics is both a conceptual and computational tool with which children are keen to engage. The descriptions of incompetent functioning ascribed to pupils' learning would therefore seem less a function of mathematics per se and much more to do with how mathematical topics are taught in school.

Given the enormous range of human intellectual accomplishment over phylogenetic time, it seems reasonable that by now we should have devised means to enable people (who in many other respects are autonomous learners) to become mathematically competent. The 'failure' would seem attributable to pedagogical practices. But pedagogical principles are derived from theories of learning, the preserve of psychologists. The superficially obvious conclusion to be drawn is that ineffective teaching has been perpetuated because psychological explanations of how learning occurs are inadequate. But this may be an invalid conclusion.

While it is doubtless true that different theories of learning cannot, with equal comprehensiveness, account for all learning phenomena, it is probably also true that different theories of learning do not have equal ease of applicability when they have to be translated into pedagogical principles and practice. Thus for all that the teacher wants the children in his/her care to be conceptually aware and mature in mathematical terms, the methodological route by which such understanding is achieved may not be practicable. How does the teacher fulfil parental and societal expectations of equipping tomorrow's adults with mathematical competence (in terms of a syllabus to be overtaken) and yet at the same time aspire to the ideal of supporting each individual child in a class of thirty or more through his/her individual rate of learning?

On the one hand there is the tension which exists between the claims of society and the claims for the individual child - a philosophical question which is not central to this thesis. On the other hand there is, for the teacher, the dilemma of taking account of the differences in performance of the children in the class given the pupil-teacher ratios, the minimal availability of resources and, in the specific case of mathematics, the competing demands of other curricular areas. All of these concerns, much evidenced in the researcher's twenty plus years of experience as a practising teacher, together with her reading, led the researcher to formulate the general research question:

> Is it possible to improve pupil performance in primary school mathematics by methods which take specific account of children's existing mathematical understanding?

Such a vast question needed clearer definition and much greater specificity both conceptually and methodologically:

<u>conceptually</u>

1. What was meant by 'improving pupil performance' and what was 'performance' anyway?

2. What 'methods' were to be used?

3. How was 'children's existing mathematical understanding' to be gauged?

<u>methodologically</u>

4. What aspect of primary school mathematics was to be dealt with?

5. What age range of children was to be used?

The generation of the above five questions, although slightly more specific than the original research question, did not permit easily available answers. They did, however, focus the researcher's thinking on how the research might proceed. In terms of contiguity an order began to emerge.

Because the researcher was also a full time practising teacher in primary education, it was seen (by the researcher) as prudent and pragmatic that the research begin in her own class.

It had also been decided (though not because of her research) that the researcher have a class of new entrants to the school.

School policy dictated that for primary one children there be a heavy emphasis on number work. These 'constraints', in turn, had implications for how the rest of the research was to proceed. Thus the researcher decided that the focus of her research would be addition and subtraction.

Children's 'understanding' of addition and subtraction was to be gauged by some form of testing. Such testing was to be of an oral nature and to be closely tied to the **conceptual** aspects, rather than the algorithmic aspects, of addition and subtraction. However, and additionally, in the light of the realization that the research was to be conducted with primary one children, an exploratory, initial study (see below) was decided upon to try to make some assessment of what number skills new entrants might bring to school.

The 'methods' to be used for developing addition and subtraction in primary one children were to evolve out of theoretical positions discussed earlier:

a) the significance of counting;

b) the need to help the child to relate his/her informal knowledge with conventional symbolism and representation;

c) an openness and respect for children's existing skills and strategies.

'Pupil performance' was to be taken as the behaviour displayed by the children in the arithmetical context. Such behaviour could be in the form of specific answers to test items or could be the spontaneous reactions of pupils during interaction with peers and/or the researcher. Claims for the improvement in pupil performance would be made in terms of the methods used for developing addition and subtraction in young children.

THE INITIAL STUDY

Since the age range of children participating in the research, and the focus of the research had been decided upon, the first research task was to mount the initial study. In attempting to gauge the current levels of attainment in number, amongst a small sample of primary one children, when there had been no formal teacher input, answers to the following questions were being sought:

1. Within what number domain can the individual child
    (i) recall the number names in conventional sequence
    (ii) obtain the numerosity of a collection of countables
    (iii) represent a given numerosity
    (iv) represent an obtained numerosity?
2. Is there any apparent pattern in children's performance on

such tasks?

3. How do the children approach such tasks?

4. Is there any apparent relationship between children's approach to and performance on the tasks?

5. Is there any evidence of the children's skill in adding informally and if so, is this linked with obtaining numerosities?

6. What implications can be drawn from the findings in terms of facilitating the development of formal addition and subtraction?

Context

In November, 1986, the initial study commenced. The researcher had a small class of new entrants (primary one) in a cosmopolitan school in Glasgow's west end. From August, 1986 until mid way through October, 1986, the children attended school for mornings only. During that period the children were given no formal teaching in number work. Formal teaching in this context refers to any activity in which the children might be expected to:

a) count to 10, and add and subtract within 10 and, further,

b) represent his/her counting, addition and subtraction on paper using the conventional symbolism of numerals 0 to 9 and the operator signs of +, -, and = .

However, the children had had lots of informal experience:

a) playing with water, sand and plasticine;

b) games to teach colour and shape;

c) assembling jigsaws;

d) construction activities with lego, bricks, construct-o-straws;

e) colouring, painting, drawing, scribbling;

f) group activities of 'chalk and chant' (in which the children chanted the number names to 10 and simultaneously scribed tally marks on to the blackboard);

g) music and movement, dance, drama;

h) craft activities using 'junk';

i) number rhymes, songs, nursery rhymes;

j) dramatic play such as 'dressing up', 'in the house', 'at the hairdressers', 'in the witch's cave', 'in the shop' and so on;

all of which are fairly typical in a primary one class and are thought to afford opportunities for classification and seriation, which are the basic categories of human thought (Turner, 1984).

Subjects

13 pupils out of a possible 20 were selected from the researcher's own class. The remaining 7 children had little or no control over the English language, thereby making it

impossible for the researcher to adequately communicate with them; and for this reason were excluded from the study. The 13 selected subjects could be said to be representative of a wide spread of both ethnic and socio-economic backgrounds insofar as:

a) Asian, African, West Indian and European ethnicity was represented in the school population as a whole and to a more diluted extent in each class;

b) parental occupation included holding down prominent posts in education, the arts, industry, local politics; through working in the service areas in shops, in restaurants, driving buses and so on; to being unemployed and homeless thus needing state support in the form of finance, temporary hostel-type accommodation and social-work supervision.

The age range of the subjects was as might be typically found in a primary one class. In November, 1986 the range was 5 years 0 months to 5 years 7 months, with one exceptional subject who was only 4 years 2 months. The median age was 5 years 4 months and the mean age was 5 years 2 months.

Procedure

The initial study was concerned to find answers to questions (listed above) associated with counting tasks and the possible relationship between such tasks. Each of these questions will now be considered in turn, in terms of description and analysis.

1. WITHIN WHAT NUMBER DOMAIN CAN THE INDIVIDUAL CHILD:
   (i) RECALL THE NUMBER NAMES IN CONVENTIONAL SEQUENCE
   (ii) OBTAIN THE NUMEROSITY OF A COLLECTION OF COUNTABLES
   (iii) REPRESENT A GIVEN NUMEROSITY
   (iv) REPRESENT AN OBTAINED NUMEROSITY?

Four criterion tasks were designed to elicit answers to this question.

(i) Recalling the Number Names in Conventional Sequence

In this task the subjects were required merely to demonstrate the extent to which they could recall the number names in conventional sequence (within twenty); that is how far they could chant one, two, three etc., since Gelman & Gallistel (1978) claim that very young children can have their own idiosyncratic naming sequences. Each subject was told:

> I want to hear how well you can count.

It was anticipated that at least some subjects would not be able to sustain the sequences correctly to 20, in which case such subjects would not be pressed to continue beyond the point at which they became muddled.

The cut-off number names of 5, 10, 15 and 20 were noted.

(ii) Obtaining Total Numerosities

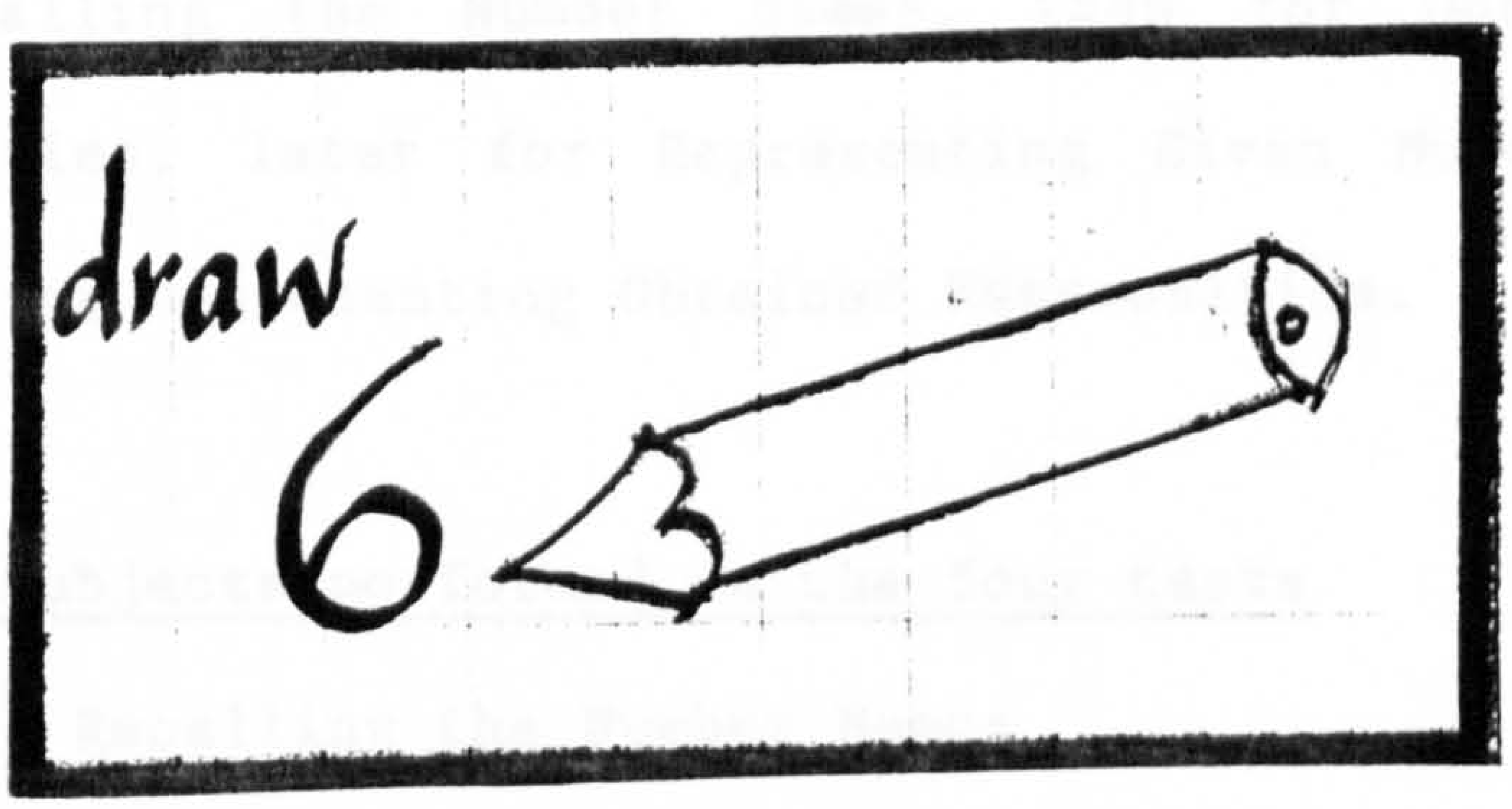
Collections of 6, 12 and 24 Logic People (see appendix 1 for a short description of Logic People) respectively were heaped in front of each subject who was each time instructed:

Find out how many people are there.

Again it was anticipated that not all subjects would comfortably manage all of the counts, in which case a subsequent presentation was not to be made by the researcher. The cut-off counts of 5, 10, 15 and 20 were noted.

(iii) Representing Given Numerosities

Each subject was given a card (see appendix 2 for the full list of cards) on which was written and drawn the instruction to draw a required number of pictures as in:



This task was repeated 10 times over several successive days and a tally was kept by the researcher of the number of correct executions for each subject.

(iv) Representing Obtained Numerosities

Subjects were given sheets of paper on which to represent obtained numerosities of pictorially represented quantities. There were 12 examples in the 0 to 5 range and 12 examples in the 6 to 10 range (see appendix 3). With the aid of felt pens the subjects were instructed to:

> Put something down to show how many things are in each set.

The counting tasks were attempted by the subjects in the order in which they have been described. Over a period of days/weeks the researcher interviewed all 13 subjects individually, first for Recalling the Number Names, then for Obtaining Total Numerosities, later for Representing Given Numerosities and finally for Representing Obtained Numerosities.

How the subjects performed on the four tasks

Table 6.1 Recalling the Number Names

Ss	to 5	to 10	to 15	to 20
1	yes	yes	yes	yes
2	yes	yes	yes	yes
3	yes	yes	yes	yes
4	yes	yes	yes	yes
5	yes	yes	yes	yes
6	yes	yes	yes	yes
7	yes	yes	yes	yes
8	yes	yes	yes	yes
9	yes	yes	yes	yes
10	yes	yes	yes	yes
11	yes	yes	yes	yes
12	yes	yes	yes	yes
13	yes	yes	yes	no
total	13	13	13	12

Table 6.2 Obtaining Total Numerosities

Ss	to 5	to 10	to 15	to 20
1	yes	yes	yes	yes
2	yes	yes	yes	yes
3	yes	yes	yes	yes
4	yes	yes	yes	yes
5	yes	yes	yes	yes
6	yes	yes	yes	yes
7	yes	yes	yes	yes
8	yes	yes	yes	yes
9	yes	yes	yes	yes
10	yes	yes	yes	yes
11	yes	yes	yes	yes
12	yes	yes	yes	no
13	yes	yes	no	no
total	13	13	12	11

Table 6.3 Representing Given Numerosities

Ss	the number of correct executions out of 10
1	10
2	10
3	10
4	10
5	10
6	10
7	10
8	10
9	10
10	10
11	10
12	10
13	10
total	130

Table 6.4 Representing Obtained Numerosities
(the number of correct executions out of 24)

Ss	0-5	6-10	total
1	12	12	24
2	12	11	23
3	12	11	23
4	12	11	23
5	12	11	23
6	11	11	22
7	12	11	23
8	12	11	23
9	12	12	24
10	12	10	22
11	12	12	24
12	12	10	22
13	12	12	24
total	155	145	300

Table 6.5 Distribution of Tally Marks and Cipherised Numerals
(in Representing Obtained Numerosities)

	0-5		6-10	
	Tallies/Ciphers		Tallies/Ciphers	
Ss				
1	10	2	10	2
2	9	3	1	11
3	10	2	0	12
4	2	10	6	6
5	10	2	6	6
6	10	2	12	0
7	7	5	12	0
8	10	2	12	0
9	10	2	12	0
10	10	2	12	0
11	6	6	0	12
12	6	6	11	1
13	2	10	1	11
total	102	54	95	61

2. IS THERE ANY APPARENT PATTERN IN CHILDREN'S PERFORMANCE ON SUCH TASKS?

As perusal of tables 6.1 to 6.4 shows, the subjects performed to a high degree of accuracy.

Twelve out of the thirteen subjects performed with complete efficiency in Recalling the Number Names in Conventional Sequence as far as 20, and the subject who did not quite achieve this standard could nevertheless accurately produce the sequence as far as 15.

In Obtaining Total Numerosities the subjects were similarly efficient with all but two managing to count an array of 20 and one of these two getting muddled when counting an array of 15.

Nor was Representing Given Numerosities troublesome to the subjects. Each demonstrated complete mastery.

And finally, in Representing Obtained Numerosities the subjects were very accurate. No subject made more than two errors and the errors were + or - 1 of the cardinal number.

It is fully conceded that recalling number names in sequence tells us nothing about the ability to count, a point made by Fuson & Hall (1983). Accurately produced number sequences merely facilitate communication between persons when counting proper is involved.

Counting performance involves two complementary behaviours. Firstly there is the behaviour of obtaining a total numerosity and secondly there is the behaviour of representing a given numerosity. Real life examples of obtaining a numerosity would include finding out how many people are in the room, calculating the amount of money in one's wallet and reckoning the number of miles for a particular journey. Real life examples of representing a given numerosity would involve choosing six apples, serving four bowls of soup or writing two letters.

In reality these behaviours frequently have to be co-ordinated: one has to obtain a numerosity <u>and</u> represent it, so one has to make one's preparation for a journey and then make the journey, or one has to estimate the number of guests being invited to the party and prepare food accordingly!

It was this perceived analysis of what constitutes counting performance which underpinned the design of the tasks which were administered to the subjects. If these tasks were tapping the underlying competence of counting, it seems reasonable to conclude, at least, tentatively, that counting for these subjects was well established.

In the absence of disparate results amongst the subjects, their attempts to represent obtained numerosities was perhaps the most interesting aspect of their counting performance.

3. HOW DO THE CHILDREN APPROACH SUCH TASKS?

Generally speaking, the subjects tackled the tasks with confidence. There was no point at which any subject seemed perplexed or even unsure about what to do. From this one may conclude that the instructions to the subjects were specific and unambiguous.

In Obtaining Numerosities different types of behaviour were observed amongst subjects. Some, of their own volition, brought a sense of order out of the heap of Logic People presented to them. They arranged the people in a line and either conducted the count after lining up the people, or lined up and counted the people simultaneously. Other subjects, however, tried to count the heap of Logic People as it was - in disarray - with the result that they could not distinguish between what had, and what had not, been counted. To these subjects the researcher suggested that it might help to make a line of Logic People. This the subjects did and then counted from one end of the line.

Pointing to, and touching, the Logic People was much in evidence while the subjects were conducting their counts. Gelman & Gallistel (1978) suggest that "pointing behaviour seems to be central to the counting procedure".

Some subjects counted aloud while others counted silently (in which case the counting was deduced from the pointing/touching behaviours) and finally announced the cardinal number. Silent counting is regarded by Gelman & Gallistel (1978) as a developmental advance on counting aloud.

These different behaviours appeared to cluster in patterns:
firstly, the subjects who needed the prompt from the researcher to line up the people invariably counted aloud;
secondly, of the subjects who spontaneously lined up the people and then proceeded to count, some counted aloud whilst others counted silently;
thirdly, the subjects who simultaneously lined up and counted, tended to count silently.

These clusters of counting behaviour would seem to suggest a development in sophistication from the raw state where the subject can deal with only one process at a time (that is, organizing the entities to be counted and then counting, aloud) to the more refined state where the subject can deal with more than one process at a time (that is, simultaneously organizing and counting, silently) with an intermediate state where the subject uses behaviours from each of the two extremes.

In Representing Given Numerosities, the only comment to be made is about the meticulous way in which the subjects addressed themselves to the task. For the larger numerosities (that is, beyond 4) subjects were frequently to be seen checking how many pictures they had drawn. For example, when required to draw 7 cups the subjects could be seen counting when they had drawn only 5 cups and later when they had drawn 6 cups and finally when they had drawn 7 cups. The subjects had not been instructed to carry out such checks but since they all did so, the strategy was presumably, one of their own devising and one which had meaning for them.

In Representing Obtained Numerosities there was variety in the subjects' approach. Some grappled with conventional notation. In her capacity as a teacher, the researcher had just begun teaching numeral formation. All of the subjects, however, used an amalgam of cipherised numerals and tally marks in representing their counting. Representations of zero ranged from the conventional nought to a dot, or sometimes the paper was deliberately left blank; the latter forms of representation being interpreted by the researcher as being part of the tally mark system.

The researcher asked the subjects why they had used 'strokes' (the children's referent for tally marks). The typical answer was that "strokes are easier to make than numbers". Indeed some of the subjects were heard to complain, whilst executing this task, that they "couldn't make" particular numerical symbols;

three, five and eight being particularly problematic. The real explanation for this vacillation between the two systems is unclear. It may be, as the subjects themselves said, something to do with the manual problem of forming the numerals. On the other hand, the preference for tally marks may have had its roots in greater conceptual satisfaction for the subjects in making marks in one-to-one correspondence with the items to be counted. Theoretical considerations of tally marks will be discussed below.

4. IS THERE ANY APPARENT RELATIONSHIP BETWEEN CHILDREN'S APPROACH TO AND PERFORMANCE ON THE TASKS?

Counting performance, it will be recalled, was earlier delineated as comprising both the **obtaining of numerosities** and the **representation of given numerosities** in the co-ordinated activity of **representing obtained numerosities.** Mastery of both the subordinate skills and the superordinate skill was evidenced in all of the subjects.

Counting competence, on the other hand, cannot be directly observed and is, according to Gelman & Gallistel (1978) an integrated, sophisticated system comprising five counting principles:

a) <u>the one-one principle</u>: attaching a different number name to one and only one of the entities to be counted;

b) the stable-order principle: applying the correct number names in sequence;

c) the cardinal principle: asserting the numerosity of a set from the number name applied to the last member of the collection to be counted;

d) the abstraction principle: addressing the numerosity of a set and not being distracted by perceptual differences;

e) the order-irrelevance principle: counting effectively even if the order of enumeration is altered.

It is therefore appropriate to consider the subjects' performance in the light of Gelman's counting principles to locate the descriptive, empirical data from this initial study in some sort of theoretical framework.

In terms of the one-one principle, all subjects, within the limits of the counting tasks were able to attach a different number name to one and only one of the Logic People; albeit that some subjects needed a prompt to line up the people for counting.

In terms of the stable-order principle, all subjects were able to apply the correct number names in sequence.

In terms of the cardinal principle, the subjects were able to assert the numerosity of the set from the number name applied to the last member of the collection to be counted. They responded to the reminder question, 'So how many people are there?', by simply repeating the name applied to the last countable. None demonstrated the phenomenon of responding to the reminder question by starting to count all over again which, in its manifestation, is, for Gelman & Gallistel (1978) an indicator that the child has not yet grasped the cardinal principle.

In terms of the abstraction principle, all of the subjects addressed themselves to the numerosities of the collections and were not distracted by the perceptual differences of the Logic People: that is green, red, blue and yellow men, women, boys and girls who were walking, sitting or standing.

In terms of the order-irrelevance principle, the subjects could count collections of 6 and 12 from any given point indicated by the researcher: such as from the middle, third from the right, fourth from the left and so on. That the subjects could remember the exact Logic Person from which they had started counting was perhaps facilitated by the variety of colours and shapes of the Logic People. This might not have held had the stimuli been all brown Unifix Cubes, for example. When required to count collections of 24 from anywhere but one end of the collection, the subjects got somewhat muddled. While Gelman & Gallistel (1978) argue that a grasp of the order-irrelevance principle subsumes a knowledge of the other four counting principles, this distinction in performance between being able to count 12 and not being able to count 24 when the order of enumeration is altered would suggest that the counting principles develop in some concerted way and are applied to increasingly larger number domains, rather than developing in some discrete fashion.

The strategy of repeatedly counting in Representing Given Numerosities would, further, seem indicative of Gelman & Gallistel's (1978) claim that children, having grasped the how-to-count principles (that is, the one-one principle, the stable-order principle and the cardinal principle), are motivated to routinize their counting in what Gelman & Gallistel regard as a truly Piagetian schema, a point of reference against which the individual checks his/her understanding of the world.

In Piagetian parlance the schemata assimilate new situations/experiences and accommodate to them due to the basic need to reconcile imbalance between the cognitive structures and the environment. Translated into the neo-Piagetian terms and content of this thesis, the counting schema is an organizational framework which allows the child to evaluate and refine his/her counting behaviour.

The lining up of countables in Obtaining Numerosities seems indicative of both the abstraction principle and the order-irrelevance principle:
of the abstraction principle because the 'lining up' took no account of the varying sizes, colours or shapes of the countables when the opportunity was (albeit implicitly) inherent in the task to firstly sort according to some criterion;
of the order-irrelevance principle in that the subjects appreciated that each member of the array had to be counted, and counted only once - in this the subjects (many of them spontaneously but some with support from the researcher) demonstrated their knowledge that number names are temporary designations only.

The task of Representing Obtained Numerosities was devised out of a desire to explore what Hughes (1986) sees as a "serious mismatch between the system of symbols which children are required to learn, and their own spontaneous conceptualizations". Hughes argues:

> that although they see written numerals around them all the time, children in most Western societies are not usually introduced to written arithmetic until they start school. The exact age at which this takes place varies from country to country but, whenever it happens, the basic problem is the same: children must learn to link the new written form of representation with the concrete understanding of number which they already have when they start school.

Prior to carrying out the task of Representing Obtained Numerosities, the subjects already knew something of "the system of symbols" in that they could recognize some of the conventional numerical symbolism (when carrying out the Representing Given Numerosities task). However, they had had no opportunity to demonstrate "their own spontaneous conceptualizations" of how to represent a numerosity because in the earlier representational work, the subjects had been specifically instructed to make pictographic representations.

In the event, the subjects seemed well aware of two systems for representing their counting, albeit that neither form of representation had been completely accessed by the subjects.

Evidence for the subjects' awareness of conventional symbolism is to be found in the observations that:
(i) all of the subjects had perfectly represented numerosities pictorially, when the stimulus for this had been supplied in conventional form (see Representing Given Numerosities task);
(ii) none of the subjects made pictographic representations in the Representing Obtained Numerosities task. Although this would have been odd given that the stimuli for this task were in pictographic form, it would not have been unreasonable since the subjects had not been specifically directed as to <u>how</u> to make their representations.
These observations would seem to suggest that the subjects <u>knew</u>

that their represented response had to be a translation from or to a different level of abstraction. This would be in line with Hughes's (1986) view that young children do conceptualize some sort of representation of a stimulus which is at a different level of abstraction.

The subjects' attempts to represent obtained numerosities in a conventional manner could be said to be "spontaneous" insofar as the subjects had not been instructed as to how the representation was to be made. The subjects' spontaneity, however, included the use of tally marks. Hughes (1986) refers to the phylogenetic origins of tally systems and raises the possibility that in using tally marks children are using a fundamental and universal method of representation. He points out that the physical action involved in making a downward stroke "is very close to the action of reaching out and touching objects when counting them: they are both ways of 'marking off' which seems to be very fundamental". Moreover, Hughes (1986) found in his own work that in representing quantity "children themselves tend to use methods based on one to one correspondence".

Given that the formation of cipherised numerals was problematic to many of the subjects, and given the relative ease with which tally marks are made, it is perhaps surprising that the subjects did not use the tally system to the exclusion of conventional notation. And yet the subjects' use of the tally system was not necessarily perfect. In representing zero, for example, some subjects chose to make no mark; justifying this on the grounds that 'there was nothing there'. These subjects seemed not to see the need to 'acknowledge' that 'there was nothing there'.

A tentative explanation for the subjects' use of both conventional notation and the tally system (neither of which was used exclusively or perfectly) seems to be that while the subjects appreciate the existence of, and are keen to adopt, conventional notation for representing quantities they are nevertheless stymied by their own lack of manual dexterity in forming cipherised numerals; in which case they resolve their frustration by 'reverting' to the more secure, less abstract system of making tally marks. That the subjects are possibly conceptualizing not one but two imperfectly developed representational systems and using both spontaneously is a strength rather than a weakness.

5. IS THERE ANY EVIDENCE OF THE CHILDREN'S SKILL IN ADDING INFORMALLY AND IF SO, IS THIS LINKED WITH OBTAINING NUMEROSITIES?

<u>The Adding Bingo Game</u> The subjects were in groups of not more than 5. Each subject was given one of the following cards.

Figure 6.1

7	6	3
5	9	11

2	3	7
4	10	5

3	10	12
4	8	9

6	12	8
9	11	7

10	7	12
8	4	5

Each card (measuring approximately 10cm by 5cm) had 6 cells, with each cell being represented by a number in conventional symbolism. Two conventional dice, the faces of each showing dots for numerosities 1 to 6, were shaken together. The dots thrown were summed and if the total matched a numerosity on the card, a plastic counter was placed on top of the numeral. The first child to completely cover his/her card was the winner of the game.

To begin with, the researcher prompted each subject after he/she had thrown the dice by asking:

How many dots are there altogether?

The context of the game, with its conventions of turn-taking and of there being a winner, was one with which the subjects were very familiar, and learning the particular rules of this game did not seem to present the subjects with too much difficulty.

Initial reminders by the researcher to sum the dots were quickly dispensed with. Most of the time the subjects found the sum by counting, using the 'counting all' strategy. Finger pointing and head nodding showed this. A few of the subjects knew some of the smaller number bonds such as 1 and 1 are 2. One subject was able to derive a number fact from another known fact. When asked how he knew that 5 and 5 are 10 he replied, "cause 4 and 4 are 8 and another 2 is 10". Yet another subject consistently and correctly knew all of the number bonds. When asked how he knew, his reply was, "cause I just know".

Because counting/adding was an intrinsic part of the game, the subjects were constantly getting practice in the task without its seeming a chore or without its being imposed externally. Hughes (1986) makes a similar point when discussing the intrinsic motivation of games as a means by which number proficiency can be increased.

There is, however, another advantage in using the games context which if it is not of immediate concern to the researcher it most certainly is to the teacher. And that is the advantage of keeping the children in the mathematical situation without the teacher necessarily having to be present. Floyd et al (1982) spell out this point when they say:

> many teachers reject the idea of producing practical learning experiences because of the worry that they will be tied down to being present throughout the time that the activity is going on, which is completely infeasible with any class of normal size.

Floyd et al recommend that teachers "try and build the activity into a game where the winning strategy involves the very process you are trying to teach".

The Adding Bingo Game incorporated the advantages listed above and by providing additional cards, dice and plastic counters, became a game which many of the children in the researcher's class chose to play, thus freeing the researcher, in her capacity as a teacher, to attend to other groups of children.

The purpose of the Adding Bingo Game was to explore the informal addition skills of children who had had no formal teaching in addition. As Gelman & Gallistel (1978) and others, have established, counting is a fundamental activity from which informal addition appears to stem almost naturally; especially so when the counting involved employs the 'counting all' strategy. Nothing in the Adding Bingo Game denied that informal addition, when effected by the 'counting all' strategy, was a consequence of being able to obtain a numerosity.

6. WHAT IMPLICATIONS CAN BE DRAWN FROM THE FINDINGS IN TERMS OF FACILITATING THE DEVELOPMENT OF FORMAL ADDITION AND SUBTRACTION?

The purpose of this initial study had been to assess what number attainments primary one children had on entering school and to glean, from careful and controlled observation, factors which ought to be taken into account when formal number work commenced. So what was learned?

Children are remarkably knowledgeable about our number system. They know it is a means through which one can make observations of one's environment. They also know that such observations can be recorded in different forms. They are nevertheless somewhat confused about the highly systematized nature of our number system. Through an inability (perhaps) to monitor their own cognitive behaviour they may not appreciate, for example, that it is prudent to spatially arrange the members of a disorganized array in some sort of order before executing a count. Similarly in their desire to be participating, functioning, 'doing' people, they do not yet appreciate the need for consistency in representing their observations; that in the real world adults do not usually vacillate between the tally system and conventional notation for representing their mathematical behaviour.

Perhaps the most important implication to be drawn for early maths teaching is that we as teachers use as many intermediate steps as need be to help the child make links between his/her informal knowledge and regularized, conventional formalism. This means:

1. helping the child to make explicit to self and others what knowledge he/she <u>does</u> have: if the child wants to use tally marks, fine, but this has to be recognized as a different system from conventional notation;

2. building on what the child 'gives': if the child can

represent his/her counting pictorially then he/she can be assisted to represent addition and subtraction pictorially;

3. removing unnecessary obstacles from the child's learning: if the formation of cipherised numerals is problematic, the learning of such a skill should be kept distinct from the teaching of number - plastic, magnetized numerals are readily available for children to manipulate instead;

4. generally being 'open' to the child's contribution: as teachers we must believe that the child really is trying to make sense of his/her environment and not trying to sabotage our attempts to teach.

Since formal symbolic notation is an intrinsic part of mathematical representation and since skill in such representation has not become routinized for the young child, it is essential that teachers make every effort to mesh their 'expert' knowledge with the child's 'inexpert' knowledge. Not to do so and then claim that a child's number calculation is wrong is to make a judgement about the child's cognitive competence on the basis of psychomotor performance. That children come to school with a competence in counting is not in question. That they do, however, does not allow us to assume that they are conversant with all of its symbolism and conventions.

How generalizable are the findings likely to be?

In carrying out research in the field of Social Science there is the desire on the part of the researcher to make generalized claims from his/her always limited experience to the wider social world. Indeed it can be argued that given the impossible logistics of describing, let alone explaining, the criterion behaviour(s) of the population at large, the whole point of carrying out research is to go beyond the information available from a small part and make inferences to the whole, in some cautious, precise and reliable fashion.

To begin to make generalized claims for the results in this initial study firstly requires that the sample of subjects mirrors a population of primary one children. Was the sample truly representative of at least a Glasgow if not a Scottish population? Did each member of the population have an equal chance of appearing in the sample?

It will be recalled that the sample used totalled 13 persons, a tiny number. However, given that the sample was drawn from a class which was one of three infant reception classes in one of the largest primary schools in Glasgow; given that children were assigned by the school to particular classes to counterbalance sex, age and ethnicity; and given that the school was commonly recognized by both politicians and educationalists as being a microcosm of society in its ethnic and socio-economic make up, it is possible to claim that the sample is not obviously unrepresentative.

But even if the sample is representative, the importance of the data gathered depends on how safely one can generalize from them, on the extent to which one can claim that the scoring by the subjects would truly describe the population. The quantification of the probability of error in such estimating is a matter for statistical analyses.

But perusal of the data in tables 6.1, 6.2, 6.3, 6.4 and 6.5 shows the distribution of scores to be very skewed. Since the absence of a normal distribution in subjects' scores violates one of the assumptions for using parametric tests, it is clearly inappropriate to do so. This then leaves the question of whether there is anything to be learned from using non parametric tests. Non parametric tests mostly depend on the rank ordering of data to highlight significant differences or correlations. But since the differences between scores in this initial study are almost non existent it seems inappropriate to use non parametric tests. Statistical analyses seem irrelevant when it is recalled that:
in Obtaining Total Numerosities all thirteen subjects could count to 10, twelve subjects could count to 15 and eleven subjects could count to 20;

in Representing Given Numerosities all thirteen subjects achieved 100% accuracy;
in Representing Obtained Numerosities all subjects scored at least 22 out of 24 items correctly.

Admittedly, however, more subjects made errors in Representing Obtained Numerosities. Only four subjects scored all 24 items correctly while the remaining nine subjects made one or two errors. It is interesting to speculate on why the majority of subjects in the Representing Obtained Numerosities task should have made errors, as this contrasts with their respective performances on the other tasks.

There may be an explanation in that in Representing Obtained Numerosities the subjects had to cope with a number of variables:
a) remembering the obtained numerosity
b) deciding whether to represent the numerosity by tally marks or cipherised numerals
c) struggling with the mechanics of forming numerals when using conventional symbolism.

By way of contrast, when being required only to Obtain Numerosities, the subject could announce the cardinal number as soon as he/she had finished counting. Similarly, when being required to Represent Given Numerosities, the subject had a point of reference. The subjects had the card with specific instructions as to how to make the representation in front of him/her for as long as it was needed. In both of these tasks the subject's attention was focussed on only <u>one</u> facet of counting. But in Representing Obtained Numerosities, the subject's attention had to be given to <u>both</u> facets <u>and</u> to making a decision as to what form the representation should take. Given that the subject's short term memory trace of the obtained numerosity would fade in seconds (Baddeley, 1976; Gregg, 1975) which could mean that the subject had to obtain the numerosity several times (in other words recount, perhaps more than once) in the execution of one example; and given that the subject had choice in how to make his/her representation; and given that the formation of cipherised numerals was not yet routinized for the subject; it is reasonable to deduce (even if in an understated way!) that the subject's processing capacity was heavily loaded.

And yet representing an obtained numerosity is at the very heart of all we require children to do in formal number work. Children are expected to count, add, subtract, multiply and to divide, and to record this activity not only once a day in the course of school mathematics, but for as many times as there are examples provided for the children to work through.

As has been suggested already in this chapter, the most interesting aspect of the subjects' counting behaviour seems to have been revealed in Representing Obtained Numerosities insofar as it was on this task that subjects manifested greatest variability in execution. Are the differences, then, in any way real? Specifically, did the subjects have a personal preference in using the tally system or conventional notation? Furthermore did the magnitude of the numerosity affect the form of representation used? Wilcoxon tests were run on the distribution of tally marks and cipherised numerals in both the 0-5 range and the 6-10 range. Subjects used tally marks significantly more often in the 0-5 range (p.<.1, two tailed) than they did cipherised numerals. For larger numbers, however, differences were such that they could be due to chance alone. Given the weak level of significance in preference for tally marks in the lower magnitude; and given the lack of significance in difference between tally marks and cipherised numerals in the higher number magnitude, it cannot be claimed that the distributions of representation would be found either in another sample or in a parent population.

Even if, however, the findings are not generalizable, the evidence from this small sample is probably as clear as it can be: that the subjects are sufficiently proficient in counting to be able to proceed to formal addition and subtraction. The issue now is whether or not a means for teaching formal addition and subtraction can be found such that both processes are meaningful to the children. The next chapter describes this researcher's attempts to do just that.

In summary:

(i) young children, on entering the formal school system, are able to obtain numerosities and represent given numerosities very competently;

(ii) the same children find it more difficult to combine these tasks into representing obtained numerosities, a task which is very characteristic of school mathematics;

(iii) informal addition is an extension of being able to obtain a numerosity;

(iv) the facility to represent obtained numerosities is more problematic: this may be due to the psychomotor 'newness' of using conventional notation or it may be due to the conceptual confusion of not knowing what form of representation to adopt;

(v) as teachers we need to be aware of the previous point and be prepared to <u>support</u> children through their transitions from informal to formal characterization of the task in hand.

ADDENDUM TO CHAPTER 6

During the intervening months between November 1986 and March 1987, excluding an extended Christmas holiday on account of inclement weather, the subjects had practised their counting skill on a daily basis: they had obtained numerosities, they had represented given numerosities and they had represented obtained numerosities. Many had learned to play number games such as Snakes and Ladders, Ludo and Dominoes; and they had continued to practise and improve their skills in the writing of cipherised numerals.

The children's attainment in Recalling the Number Names in Conventional Sequence was also considerable. In March 1987 they demonstrated their chanting skills thus:

Table 6.7 Recalling the Number Names

Ss	last cardinal name stated
1	99
2	100
3	123
4	98
5	100
6	100
7	79
8	49
9	39
10	69
11	49
12	39
13	29

Subjects 2, 3, 5 and 6 were stopped by the researcher. They could probably have gone on for longer but they were clearly tired by their efforts. Perusal of the last number name stated, in table 6.7, shows that 8 out of 13 of the subjects stopped their chanting on a '9', the last number name before a new decade. When prompted by the researcher with the new decade name the subjects were able to continue ..1, ..2, ..3, until ..9, when they became unstuck at the subsequent decade name. This phenomenon is noted also by Fuson & Hall (1983) who maintain that:

> word sequences produced by children aged 4½-6 years of age indicated that they knew the repeating one to nine pattern in the decades (eg thirty, thirty-one . . . ., thirty-nine) but that they had not yet solved the 'decade problem' (that is they did not produce the decades in the right order).

The subjects in this study would appear to have considerable control over the conventional number naming sequence. Fuson & Hall (1983) maintain that "the age at which the whole sequence to 100 is acquired seem heavily dependent upon the practices of individual teachers as well as on subject variables". They go on to claim that their observations in suburban schools suggest that "this can be accomplished by the end of kindergarten for most middle-class children if teachers provide moderate amounts of sequence production activities".

If it is remembered that American children start formal schooling at 6 years of age, a year later than their Scottish counterparts, then kindergarten children will be the same age as primary one children in the Scottish system.

In this study the amount of teacher input into the generation of the cardinal number names had been very little. The teacher counted out loud the number of children present in the morning and afternoon but this certainly could not account for the chanting skills of the subjects.

Gelman & Gallistel (1978) argue, however, that it is "the development of the child's ability to perceive underlying generative rules" which "leads to the mastery of a count sequence that requires a limited amount of rote learning but is capable of being extended indefinitely". In other words the child learns by rote the one to nine pattern and probably also the ten to fifteen pattern, although "fourteen is the first English count word whose derivation from an earlier member of the rote sequence is completely transparent" (Gelman & Gallistel, 1978). Thereafter the child learns the decade names: twenty, thirty, forty (and here again perception of their derivatives may enable the child to master the subsequent decade names of fifty, sixty, seventy, etc); at which point "all other count words can be derived from the application of the generative rules embodied in the already mastered count words" (Gelman & Gallistel, 1978).

The subjects' intrinsic motivation to chant the number names and to apply their counting abilities in their daily living experiences was evidenced in the following types of informal, social interaction which the subjects themselves initiated.

<u>Example 1</u> At lunch time when about half the class departs to the dining room for school lunch, the remainder gets ready to go home for lunch. Any one of the remaining children can announce how many people are left in the room. Sometimes the teacher is included as a countable, sometimes not; although she is usually told if she has been included. The count is usually correct, give or take one.

<u>Example 2</u> The children are beginning to gather round the teacher for a general instruction or a whole class activity such as a story or singing. Those who arrive early do one or both of the following:
a) announce who has arrived first, second, third and so on as in 'I'm the first, Jean was second, Bob was third', although by the time the sixth or seventh person has arrived he/she is referred to by the cardinal rather than the ordinal number as in 'Andrew was number six and Peter was number seven';
b) count, in a cardinal context, the number of children who have arrived beside the teacher.

<u>Example 3</u> The teacher claims that she does not know how many people are in the class. A child offers to help by counting the people present. The offer is gratefully accepted and the child counts the children who are sitting as a group round the teacher. After a first count the child is requested, by the teacher, to count again to check. Usually a second count produces a different answer. The discrepancy in the count is pointed out to the child who is then asked what he/she is going to do about it. The less efficient counters proceed to try to count yet again the muddled mass of bodies sitting on the floor. More efficient counters, however, suggest that everybody should sit in their own seat or that everybody should stand up in a straight line. The assembled body invariably acquiesces to the counter's request, amid reminders of 'Remember to count yourself'. Such an organizational strategy enables the counter to produce the correct count.

<u>Example 4</u> At milktime two children are give joint responsibility for giving out the cartons of milk and straws. They are asked by the teacher, 'Will there be enough cartons of milk?' The milk cartons and all of the children are then counted by the helpers whereupon the teacher is reassured that because there are more cartons of milk than people in the class, there will be enough milk. Sometimes the teacher is even told that there will be two or three or four cartons left over because there are not enough children for all of the cartons of milk. When the helpers are asked by the teacher, 'Will there be enough straws?' a similar counting procedure is effected by the helpers who sometimes have to tell the teacher that there are not enough straws. Again she may be specifically told how many more straws are needed.

Example 5 The children are lining up at the classroom door to leave. Those at the head of the queue start chanting a rhyme which has subsequently been learned by many others in the class, and by the teacher. The rhyme is as follows:

> first the worst, second the best,
> third the royal princess;
> fourth the ghost, eating toast,
> halfway up a lamp post.

The rhyme had been introduced by one little girl who had learned it from her mother. Another little boy had learned a similar rhyme from his older sister. This rhyme makes greater use of ordinal number:

> first the worst, second the best,
> third the dirty donkey;
> fourth the king,
> fifth the queen,
> sixth the royal jelly bean.

These informal observations exemplify some of the empirical findings noted by eminent researchers in the neo-Piagetian mould:

firstly, that children learn number words in a variety of different social/linguistic contexts (Fuson & Hall, 1983);

secondly, that children do want to apply their counting abilities to many situations (Gelman & Gallistel, 1978);

thirdly, and perhaps most controversially, that the child's knowledge of numerical equivalence comes from obtaining a numerosity as distinct from constructing a physical correspondence (Gelman & Gallistel, 1978).

This last point needs some explanation.

The researcher was, quite understandably, excited by the counting accomplishments of her young subjects who were also the pupils in her class. As is normal practice in schools, the researcher in her capacity as a teacher would discuss with colleagues the children's progress. During such discussion it was pointed out time and again to the researcher by her teaching colleagues that 'before children can count properly they must have one-to-one correspondence so before you teach children to count you must do all the pre-number work of matching, sorting, ordering and pairing'!!. This teacher had not, at least in any conscious or structured way, dealt with the 'pre-number work' and yet the children in her class appeared to be counting well.

The strength of belief in others, of one-to-one correspondence being a necessary condition for counting forced the researcher to return to the literature.

The literature on children's number accomplishments uses the term, one-to-one correspondence freely but rarely in any precise way, with the result that the reader may construe from the term a meaning not intended by the writer. One-to-one correspondence, for example,

> "means there is an 'exact match' between two sets" and the "practical experience of matching is a preparation for counting" (Deboys & Pitt, 1979);
>
> "is basic to the concept of number and it is much simpler than counting" (Liebeck, 1984);

"is a foundational concept of mathematics" (Klein & Starkey, 1987).

What does all this mean?

The following attempt to clarify what is meant by one-to-one correspondence owes much to Klein & Starkey (1987).

One-to-one correspondence is, essentially, abstract knowledge. One-to-one correspondence is a hypothetical construct, the existence of which is inferred from certain types of behaviours in the real, physical world. One-to-one correspondence is not of itself a directly observable activity. There are, however, two classes of behaviour (in the context of number performance) which allow one to deduce that the person or persons involved in the behaviour is/are in possession of the knowledge of one-to-one correspondence.

Class 1 Correspondence Construction

Correspondence construction is the pairing or mapping of every member of a set with or onto one and only one member of a second set. If this pairing is 'perfect' the sets are equivalent but if the pairing is not 'perfect' (though accurately executed), the sets are non equivalent.

Class 2 Counting

Counting is the mapping of a consistent list of number names onto a set of objects. The sequence of the number names may be conventional or idiosyncratic. A unique number name is assigned to each object in the set and the final number name that is used represents the cardinal value or numerosity of the set.

Largely because children of less than 7 years of age typically fail the classic, Piagetian conservation of number test (the reason for this, it is argued, being principally because of a failure to grasp the principle of one-to-one correspondence) it is reasoned that an understanding of the principle of one-to-one correspondence is a necessary prerequisite to counting. In Piagetian theory, one-to-one correspondence "is the psychologically primitive basis for a judgement of numerical equality" (Gelman & Gallistel, 1978).

Here again, however, what is meant by one-to-one correspondence is not made clear. Yes, of course, a knowledge of one-to-one correspondence in the counting sense (as referred to in class 2, above) is required if one is going to count, but it does not follow that in order to count effectively one must necessarily be able to effect a correspondence construction (as referred to in class 1, above).

The research, in fact, (Gelman & Gallistel, 1978; Langer, 1980 and 1986; Gopnik, 1981; Sugarman, 1983) indicates that the behaviours of constructing a correspondence and counting both begin to develop in the infant's second year of life. It does not therfore seem that one 'grows' out of the other. Furthermore Klein & Starkey (1987) have found that in young (between 4 and 6 years of age) children's arithmetical reasoning, the children's explanations for the outcome of the operations of addition and subtraction were heavily influenced by the type of behavioural scenario to which they had been exposed. When asked to make a judgement on the basis of counting the members of sets, the subjects referred to the cardinal values of the collections but when asked to make judgements on the basis of constructing correspondences between sets the subjects referred to the relationship between the members in the two sets.

The concept of one-to-one correspondence must not therefore be conflated with correspondence construction. While correspondence construction implies a knowledge of one-to-one correspondence, so also does counting. The two behaviours of counting and correspondence construction are manifestly different even if they do grow from the one root of one-to-one correspondence.

Gelman & Gallistel (1978) argue that while "at some point in the individual's development he recognizes that sets that can be placed in one-to-one correspondence are equal", this is "a later stage in the use of reasoning principles". The very young child's criterion for deciding whether two sets are numerically equal is to count them and see, and on the basis of obtaining the same/different cardinal number for each set he/she will agree that the two sets are/are not equivalent.

In summary:

(i) children's counting performances develop when situations are provided which allow them to obtain and represent numerosities;

(ii) this will occur when contexts are planned or exploited; it does not have to be formal and 'heavy handed'; it can be fun and appear to be spontaneous;

(iii) one-to-one correspondence is a concept not an activity in the sense of a behavioural manifestation;

(iv) counting and constructing correspondences are two classes of manifestation of one-to-one correspondence.

THE CONTINUATION OF THE EMPIRICAL WORK:

THE MAIN STUDY

The Traditional Approach to Addition and Subtraction

Traditionally, the teaching of addition and subtraction (and for that matter, multiplication and division) has been conceived of as being relatively straightforward. After minimal attention has been given to orienting the learners to the operation under consideration, the concentration of time and energy (both by learners and teachers) has been on the execution of routines which will achieve a correct answer. Practice in such routines has been considered very important, is usually provided in graded form, from 'simple' to 'difficult', and is regarded by many teachers and learners as an end in itself. During the practising of such routines some children need the 'support' of using 'concrete materials'. But there is the suggestion of there being a certain stigma attached to the use of same. (It has been the researcher's observation that teachers consider that learners should be weaned away from such 'crutches' as quickly and as soon as possible, because these 'crutches' are 'unacceptable' in the real world! By the time the children are in the middle of their primary education, this rhetoric would seem to have been communicated to the children who, in turn, either boast about their 'superior' skills in not needing concrete materials, or try to conceal their alleged lack by surreptitious finger counting.) A small proportion of those children who succeed in executing these operational routines are deemed fit to proceed to 'problems' - contextualized versions of the operations learned. Such 'problems', which are regarded as extensions of algorithmic practice, cannot be 'taught' and must therefore remain the mystical preserve of that small minority who can divine their nebulous meaning!

The reader can be forgiven for regarding the above diatribe as cynical. Such a conception of arithmetical operations which restricts learning to 'doing sums' and reduces teaching to demonstrating (usually on the blackboard to every member of the class simultaneously) how to 'do' subtraction by the decomposition method or how to 'do' long division by the DAMSON method is totally inadequate. And yet it has been the researcher's experience that this delineation of arithmetic as a series of formal algorithms is a reality. The literature (Skemp, 1971; Ginsburg, 1977; McIntosh, 1977; Glen, 1978; Dickson et al, 1984; Liebeck, 1984; Hughes, 1986; Desforges & Cockburn, 1987) too, argues against what is seen as an almost exclusive concentration on the formalism of arithmetic on the grounds that symbol manipulation becomes an end in itself; is meaningless to most children; and lays very shaky foundations for all subsequent mathematical development.

The Justification for an Alternative Approach

Essentially put, the problem with traditional approaches to the teaching of number operations is that they allow little if any room for the conceptual underpinnings of the arithmetical skill

a) to be made explicit

or

b) to develop in any way which allows the execution of the algorithm to have meaning or reality for the learner.

If as teachers we want our pupils to be thinking people, growing towards intellectual autonomy; even if we just want our pupils to be good at mathematics, we must enable learners to develop the tools of thinking - concepts. And if conceptual development is to be at the forefront of our teaching we cannot hope that, by chance, concepts will develop. While on the one hand we cannot 'insert' concepts in another's mind as one would give another a pill or tablet, we can at least try to provide situations where concepts can begin to flower and grow. As Skemp (1971) points out, "the teacher must look far beyond the present task of the learner, and wherever possible communicate new ideas in such a way that appropriate long term schemas are formed".

In attempting to address the problem of how we can better facilitate conceptual awareness in number, there is, however, no point in throwing the baby out with the bath water. There is no suggestion intended that the formalisms of number be dispensed with: just that of themselves they are not enough. Hughes (1986) develops such a point when he says:

> What seems to be clear is that both the formal and the concrete are important, and the child who has one without the other is at a serious disadvantage. Children need help in freeing their thinking from the concrete, and formalization is essential in this process. At the same time, there is little virtue in children mastering the formal symbolism if the concrete understanding is lacking. The crucial new element introduced here is the emphasis on the links between the concrete and the formal.

This is not to suggest that making the links requires a unidirectional progression from learning the algorithms to their applications in the real world. Indeed, as has been argued elsewhere in this thesis, such an approach is not particularly beneficial to children. What is being suggested, however, is that making the links requires bidirectional translation between the conceptual and formal representational facets of addition and subtraction. While it is necessary for the child to be able to extract the symbolic notation from the practical task, it is not sufficient. The child must also be able to interpret the notation in terms of a real life mathematical task. Only then can the child be said to have conceptual understanding of the operation involved.

It was this abiding concern of the researcher's to find some means whereby addition and subtraction could be meaningfully taught (in the sense of what has been said above) which led to the development of a somewhat novel but essentially simple methodology, entitled Bidirectional Translation. It was novel in the sense that it was devised out of necessity since most teacher's handbooks are (perhaps surprisingly) particularly vague in the pedagogical support they give to teachers; and the researcher was unaware of any other teacher or researcher using this approach. It was essentially simple insofar as it was based on two teaching principles:

firstly, start from a point already identified as being one with which the learners are familiar and progress to greater complexity;

secondly, explicitly model desirable/acceptable performance thereby providing opportunities for others to observe 'correct' responses.

The Method of Bidirectional Translation

Bidirectional Translation is a means by which addition and subtraction (and possibly the other operations) can be taught. Key features of this methodology include:

(i) very finely graded steps of progression incorporating what the subjects themselves could bring to the learning situation;

(ii) alternative strategies for dealing with a given situation; for example the choice of using fingers, Unifix Cubes (concrete materials) or tally marks for finding out an answer if the number fact can not be recalled from long term memory;

(iii) verbalization of operations;

(iv) repeated translation from and to the numerical representation/hypothetical real world scenarios.

On the following pages there follows a step-by-step, serial description of the methodology both for addition and subtraction. It reads as a series of notes and aides-memoire to the teacher, which is how it was written.

Series of steps for teaching addition

Step 1   Setting the scene.

The group of children is seated round the table, each child having a stack of 10 Unifix Cubes. The children are asked to take two cubes from their stacks. A magnetic numeral '2' is displayed on the magnet board.

2

MAGNET BOARD

Attention is drawn to the two cubes in front of each child and to the numeral '2' on the board. The children are told that they are to take more cubes from their stacks and that to show on the board that they are taking more cubes, a sign is used. The children are told that the sign says 'plus' or 'add on'. The magnetic '+' is affixed to the board.

2 +

MAGNET BOARD

The children are now asked to take a further three cubes from their stacks and to sit them beside the two cubes:

```
   []
 [][]
 [][]
```

A magnetic '3' is displayed on the board.

2 + 3

MAGNET BOARD

Attention is drawn to the cubes in front of the children and to the display on the board. The children are asked to find out how many cubes they took from the stack altogether. After the total has been ascertained, the teacher explains that another sign is needed to show that something has been found out about 2+3. The children are told that the sign says 'equals' or 'makes' or 'is the same as'. The magnetic '=' is affixed to the board as is the magnetic '5'.

2 + 3 = 5

MAGNET BOARD

Attention is drawn to the cubes in front of the children and to the 'number story' on the board (two plus three equals five). The children are invited to 'read' the number story aloud. This entire procedure is repeated many more times over successive days using different number combinations within 10. Zero is introduced by instructing the children to take out 4 cubes and then take out no cubes.

Step 2 Let's Pretend.

The children are introduced to the notion that cubes can be used to represent just about anything in the real world. The teacher says to the children, "Let's pretend the cubes are bananas" or "cars" or whatever. The children are instructed to take out three bananas and then another three bananas and to find out how many bananas they have in front of them. As above, considerable practice is given, and every addition activity is accompanied by its representation in magnetic numeral form.

Step 3 Silly Stories.

The children are told to listen to a 'silly story'. While they are listening they have to take from their stack of cubes the numbers mentioned in the 'silly story'

"Mummy gave me three lollipops and four sweets".

The children are asked to show their three lollipops (whereupon each child holds up the three Unifix Cubes) and their four sweets (whereupon each child holds up the four Unifix Cubes). The teacher asks, "How many things did Mummy give me altogether?" When the total has been identified the teacher asks, "How did you find out the answer?"

Step 4 Silly Stories and Number Stories.

The teacher provides a complete 'silly story':

"There are four blue sweets and two red sweets in the bag so that makes six sweets altogether".

The children are invited to use the magnetic numerals and signs to represent the 'silly story' as a 'number story' (4+2=6). The children 'read' the 'number story' (four plus two equals six) and are required to indicate which number represents the blue sweets, which number represents the red sweets, which sign represents the operation of addition and which sign represents the outcome of the operation. Again, much practice is given in this activity.

Step 5 Number Stories and Silly Stories.

The teacher provides a complete 'number story' on the magnet board (for example 1+3=4) and the children are invited to provide a corresponding 'silly story'. Allow as many children as time allows, to provide verbal contexts for any given numerical representation.

Step 6 Drawing a story - first version.

The children are instructed that instead of telling a 'silly story' they have to draw a 'silly story' for a bit of a number story which will be provided. The instruction 'draw 2+3' is given orally and is also put on the magnet board for the children to see. Provide paper and pencils/pens/ crayons and observe what happens. When each child has pictorially represented his/her 'silly story' ask the child to explain his/her story and scribe the story in front of the child. This procedure of drawing a 'number story' is repeated regularly over successive teaching sessions.

Step 7 Drawing a story - second version.

The children are invited to draw their own 'silly story' with no numerical stimulus being provided. In other words the children are not told of how many of each sub set to draw. There is now greater need than before for the children to describe/ explain their stories to the teacher (since both the numerical components and the verbal contexts are the children's own with no constraints imposed by the teacher) who again scribes at the child's dictation.

Step 8 Strategies for finding the answer.

The children are told that they will be given a bit of a 'number story' (for example 3+4=) and that they will have to find the answer. The teacher asks the children how they will find out the answer if they do not already know. The children make various suggestions:

a) count on their fingers
b) count with cubes
c) draw pictures

each of which is positively received by the teacher who then points out that:

a) sometimes we might not have enough fingers (as when summing any numbers the total of which is greater than 10)
b) cubes are not always available
c) pictures can take a long time to draw.

The teacher the demonstrates a 'method which she sometimes uses'. Whereupon she writes 3+4= on the blackboard and sets out the appropriate number of tally marks:

3 + 4 =
||| ||||

MAGNET BOARD

The results are compared using each of the three methods - fingers, cubes and tally marks. The children are given practice in setting down 'number stories' and in using tally marks (referred to as strokes) but are reassured that each of the methods is valid and that the final choice (of which method to use) is to be theirs.

Step 9 Does it work?

Only now are the children considered ready to undertake the conventional addition exercises of adding two numbers the total of which is within 10. The children undertake this activity outwith the direct supervision of the teacher - that is when she is working with other groups of children in the class. However, to check that the earlier steps in the series have been of use to the children, random, one-to-one interviews are held between the teacher and the child when the teacher in the light of a completed exercise:
a) asks the child how he/she found the answer to a particular operation, say 6+2;
b) invites the children to provide a 'silly story' for a particular addition operation, say 5+0;
c) requests the child to peruse all the examples in the exercise and identify which 'number story' is being referred to when the teacher provides a 'silly story'.

Series of steps for teaching subtraction

Step 1 Setting the scene.

The group of children is seated round the table, each child having a stack of 10 Unifix Cubes. The children are asked to take 6 cubes from their stacks. A magnetic numeral '6' is displayed on the magnet board.

6

MAGNET BOARD

Attention is drawn to the six cubes in front of each child and to the numeral '6' on the board. The children are told that they are to take some cubes away from their set of six and that to show on the board that they are taking cubes away a sign is used. The children are told that the sign says 'minus' or 'subtract' or 'take away'. The magnetic '-' is affixed to the board.

6 -

MAGNET BOARD

The children are now asked to take two cubes away from their stack of six and to return them to the 'bank'. A magnetic '2' is placed on the board.

6 - 2

MAGNET BOARD

Attention is again drawn to the board which now displays 6-2. The children are asked to find out how many cubes they have left. After the answer has been ascertained the teacher reminds the children that a sign is needed to show that something has been found out about 6-2. The magnetic '=' is affixed to the board as is a magnetic '4'.

6 - 2 = 4

MAGNET BOARD

The children are reminded that they started off with six cubes and that they took away two of them. They are now left with four cubes in front of them. Attention is drawn to the 'number story' on the board (6-2=4) and the children are invited to 'read' the 'number story' (six minus two equals four) aloud.

The entire procedure is repeated many more times over successive days using different number combinations within ten. Zero is used by instructing the children to take out eight cubes and then take away no cubes.

Step 2 Let's Pretend.

The children are reminded that cubes can be used to represent anything in the real world. The teacher tells the children, "let's pretend the cubes are dogs" or "houses" or whatever. The children are instructed to take out five dogs and then to take three dogs away and to find out how many dogs are left. As before, considerable practice is given and every subtraction activity is accompanied by its representation in numerical form.

Step 3 Silly Stories.

The children are told to listen to the 'silly story' and to operate with the cubes accordingly:

"Mummy had four apples but she gave me one to eat".

The children are asked to show their four apples (whereupon each child holds up his/her four Unifix Cubes) and to show that one was eaten (whereupon each child demonstrates the subtraction). When the children correctly identify how many apples are left, they are asked by the teacher how they found out the answer.

Step 4 Silly Stories and Number Stories.

The teacher provides a complete 'silly story':

"Three cups were on the shelf. One of them got knocked on to the floor so that left only two cups".

The children are invited to use the magnetic numerals and signs to represent the 'silly story' as a 'number story' (3-1=2). The children 'read' the 'number story' (three minus one equals two) and are required to indicate which number represents the cups at the beginning of the story, which number represents the cup that met with the accident, which number represents the cups at the end of the story, which sign represents the operation of subtraction and which sign represents the outcome of the operation. Many such verbal contexts are provided by the teacher.

Step 5 Number Stories and Silly Stories.

The teacher provides a complete 'number story' on the magnet board (for example 7-4=3), and the children are invited to provide a corresponding 'silly story'. Allow as many children, as time allows, to provide verbal contexts for any given numerical representation.

Step 6 Drawing a story - first version.

The children are instructed that instead of telling a 'silly story' they will have to draw a 'silly story' for a bit of a 'number story' which will be provided. The instruction 'draw 5-3' is given orally and is also put on the magnet board for the children to see. Provide paper and pencils/pens/crayons and observe what happens. Some teacher intervention may be required because the children may pictorially represent the five and also the three and proceed to add rather than subtract. If this happens, still request the child to explain/describe his/her 'silly story'. If required to reflect on their own story the children may not be too happy with a story in which they have five sweets, eat three of them and then be left with eight! Ask the children how they can show on paper the initial quantity, the operation and the result. Some may suggest rubbing out the subtrahend. This is perfectly reasonable but in so doing:
a) nobody will be able to see how many things were present at the beginning of the story;
b) nobody will be able to see how many things were taken away.
By emphasizing that the minuend and the subtrahend both be visually evident, hopefully some child or children will suggest that the minuend is drawn and that the subtrahend is represented by crossing out. Considerable practice is again required.

Step 7 Drawing a story - second version.

The children are invited to draw their own 'silly story' with no numerical stimulus being provided. In other words the children are not told how many things to draw initially or how many to score out. They must, however, verbally report the context in order that it can be written down by the teacher.

Step 8 Strategies for finding the answer.

The children are told that they will be given a bit of a 'number story' (for example 7-2=) and that they will have to find the answer. The teacher asks how they will find out the answer if they do not already know. Hopefully the children will suggest one/all of the following:

a) use their fingers
b) use cubes
c) use strokes

The teacher checks that the children are able to use all of these strategies.

Step 9 Does it work?

Only now are the children considered ready to undertake conventional subtraction exercises, within ten. The children undertake this activity outwith the direct supervision of the teacher, who is meanwhile working with other groups of children. However, as with addition, random, one-to-one interviews are held between the teacher and the individual child to check if the earlier steps in the series have been of use to the child. In each interview, and in the light of a completed subtraction exercise, the teacher:

a) asks the child how he/she found the answer to a particular subtraction operation, say 8-3;

b) invites the child to provide a 'silly story' for a particular subtraction operation say 2-2;

c) requests the child to peruse all the examples in the exercise and identify which 'number story' is being referred to when the teacher provides a 'silly story'.

The Context and Subjects of The Study

By March 1987 it was evident that all of the children in the class were able to count, even those whose behaviour has not been recorded in this research. Admittedly not all could count to the same level of sophistication but a wide range of performance is to be expected amongst any collection of children who are assigned to a class on the bases of sex, social and racial mix, and age alone. The time was now considered appropriate to begin to introduce at least some of the children to addition, subtraction and their representations.

16 children were selected as subjects for this study. They were all from the researcher's own class and included these from the initial study together with another 3 children who had joined that class at different times after the official intake in August 1986. These children were selected for study because their counting performance suggested that the transition to formal addition and subtraction would be stimulating and challenging to them.

It must be said at this point that the researcher was in a particularly enviable position in respect of researcher-subject relationships. She was advantaged to a point beyond that which typically pertains in psychological, experimental investigations. The researcher had had plenty of time (more than six months) to build a relationship with the subjects, a relationship which was warm and open.

Moreover, the subjects were children who had gradually been acclimatized to this particular teacher's classroom ethos. It was an integral part of all teaching/learning that the children were expected to explain and justify themselves. Correct or socially acceptable responses were not merely accepted. They were followed up with open questions which forced the children to express their views and make their own reasoning explicit.

One example of this was that after being involved in group activities at which the teacher could not be present (as is the case in any class where a differentiated programme of learning activities is operating) the children participated in debriefing discussion to reflect on their experiences and to make their individual or corporate evaluations of them.

Another example was to be found when the children came to the teacher with a problem such as 'I don't know how to do such and such'/'the glue won't stick my model together'/'there's no purple paint', and the teacher did not offer ready solutions. The problems were left with the children with teacher responses like 'what are you going to do about such and such?'/'go and talk to your friend about your model and see what you can do about it'/'see if you can make purple paint from the colours you do have'.

Finally, the children were very used to other adults in the room:

a) parents of prospective pupils would come to see if they considered the teaching approaches suitable;

b) teachers from within and outwith the school who were unfamiliar in the ways of working with first infants would come in to the class to observe and participate;

c) a professional film recording unit spent nine weeks filming teacher techniques and child development in the teaching and learning of drama skills.

In all of this the onus was on the children to explain to visitors the class routine, the purpose of what they were doing, the lines of demarcation and so on. Constantly the children were pushed to rely on their own resources, in the process of learning new skills.

It is fair then to say that in a professional context the researcher and the subjects were comfortable and easy with each other.

The focus of The Research

Given the sample of subjects and the rationale outlined above, the general research question derived was one of whether the method of Bidirectional Translation would enable primary one children to develop the concepts of addition and subtraction. In 'operationalized' terms, answers to the following questions were sought:

1. about Bidirectional Translation

(a) in executing addition and subtraction operations, what use is made of fingers, cubes, and tally marks and is this 'suspended reality' of counting aids easily translatable to and from a realistic, everyday addition or subtraction context?

(b) can the verbal contexts supplied by the subjects be categorized according to some criterion?

(c) does the use of operator signs easily become incorporated into the numerical representation?

2. about performance

Within the number domain of 10

(a) can the children translate from a verbal context to obtain the solution to an addition or subtraction operation? (This is subsequently referred to as the utlizability of the concept.)

(b) can the children translate from a verbal context to identify an addition or subtraction operation? (This is subsequently referred to as the evocability of the concept.)

(c) is there any apparent relationship between being able to utilize and being able to evoke addition or subtraction concepts?

3. about the effects of Bidirectional Translation

(a) can the children, if required to do so, articulate how they obtained a solution when they had to utilize a concept in test performance?

(b) given a predetermined selection of classes of verbal context with which to test children's conceptualizations, is there any apparent relationship between test performance in translating from a context and earlier pre-test performance of translating to a context?

(c) what claims can be made for the methodology?

The means by which answers to the above questions were to be found

1. Answers to questions about the methodology itself were to be got from natural observations of the subjects' behaviours whilst they were being exposed to this method of teaching.

2. Answers to questions about performance were to be got from testing.

3. Answers to questions about the effects of Bidirectional Translation were to be got from theoretical analysis.

Description of The Testing

The test items required the subjects to translate from a verbal context, thereby indicating whether the subjects could utilize and evoke the concepts of addition and subtraction. Subjects were to be tested individually on two different days. On one of the days the subjects were to be tested for the evocability of addition and subtraction. On the other day the subjects were to be tested for the utilizability of addition and subtraction. The total interview time was predicted as being just under two weeks, since testing was only to be done between 11 o'clock and mid-day and it was anticipated that the researcher could comfortably test four subjects per day.

For evoking concepts the following instruction was to be given:

> I'm going to read you a silly story. Listen carefully and tell me if you should add or subtract to find the answer . . . Well done, now let's try some more.

For utilizing concepts the following instruction was to be given:

> I'm going to read you a silly story. You see if you can find out the answer and tell me what it is . . . Well done, now let's try some more.

The verbal contexts were to be read and reread to the subject. Any subsequent lapse of memory on the part of the subject was to be remedied by the researcher's reading the verbal context yet again.

To counterbalance for order effects, half of the subjects were to be tested first for evocability and the other half were to be tested first for utilizability.

The number domain used was to be within 10. In other words for addition operations, the sum of the addends was not to be greater than 10, and for subtraction operations the minuend was not to be greater than 10. Zero quantities and doubles were not to be included because they frequently cause the verbal context to come across as 'strained' and unrealistic. The number triples generated according to these criteria were thus:

(1, 2, 3)	(1, 3, 4)	(1, 4, 5)	(1, 5, 6)	(1, 6, 7)
(1, 7, 8)	(1, 8, 9)	(1, 9, 10)	(2, 3, 5)	(2, 4, 6)
(2, 5, 7)	(2, 6, 8)	(2, 7, 9)	(2, 8, 10)	(3, 4, 7)
(3, 5, 8)	(3, 6, 9)	(3, 7, 10)	(4, 5, 9)	(4, 6, 10) .

Each subject would receive each number triple only once within the total of twenty verbal contexts to be presented to him/her. Number triples were to be matched to verbal contexts randomly, so that the particular combination of number triple and verbal context would be arbitrary.

In all, twenty verbal contexts were generated by the researcher as test items. They were generated according to a taxonomy devised by Carpenter & Moser (1982) which will be discussed later. Meanwhile the complete list of test items follows immediately.

Figure 7.1 Items to test conceptualization of Addition and Subtraction

The Utilizability of Addition

Joining: Betty has (x) dollies. Granny gave her (y) more dollies. How many dollies has Betty got now?
Separating: Fiona had some carrots. She gave (x) carrots to the rabbit and now she has (y) carrots left. How many did she have to start with?
Part-Part-Whole: (x) girls and (y) boys went out to play in the playground. How many children went out to play?
Comparison: Susan has (x) hats. Mummy has (y) more hats than Susan. How many hats has Mummy got?
Equalizing: There were (x) red cars in the car park. (y) more red cars came in. Now there are the same number of red and blue cars in the car park. How many blue cars are there?

The Utilizability of Subtraction

Joining: Mr Brown has (x) shirts. How many more shirts does he need to have (z) shirts altogether?
Separating: John had (z) pencils. He gave(x) to his big brother. How many pencils does John have now?
Part-Part-Whole: Daddy has (z) saws. (x) of them are very blunt and the rest are very sharp. How many very sharp saws does he have?
Comparison: There are (z) men and (x) women standing at the bus stop. How many more men are at the bus stop?
Equalizing: There are (z) forks and (x) knives in the drawer. How many forks should I take out so that there are the same number of forks and knives in the drawer?

The Evocability of Addition

Joining: Tom has (x) bananas. Susan gave him (y) more bananas. How many bananas has Tom got now?
Separating: Neil had some toys. He gave (x) toys to his little brother and now he has (y) toys left. How many did he have to start with?
Part-Part-Whole: There are (x) girls and (y) boys in the room. How many children are in the room altogether?
Comparison: Mary has (x) cats. Christine has (y) more cats than Mary. How many cats does Christine have?
Equalizing: There were (x) boys in the playground. (y) more went out. Now there are the same number of boys and girls in the playground. How many girls are in the playground?

The Evocability of Subtraction

Joining: Karen has (x) lollipops. How many more lollipops does she need to have (z) lollipops altogether?
Separating: Imran had (z) sweets. He gave (x) to Linda. How many sweets does Imran have now?
Part-Part-Whole: Eva has (z) pens for colouring with. (x) of them are blue and the rest are red. How many red pens has Eva got?
Comparison: There are (z) girls and (x) boys in the cloakroom. How many more girls are in the cloakroom?
Equalizing: There are (z) cups and (x) saucers on the table. How many saucers should I take away so that there are the same number of cups and saucers on the table?

The order of presentation of verbal contexts was to be randomized (within conditions) for each subject. Unifix Cubes were to be available for the subjects' use when being tested for the utilizability of the concept.

It is important to acknowledge at this point that the lexical complexity of the verbal contexts was not considered. Contexts were generated according to the Carpenter & Moser (1982) format which was concerned with the underlying logical structure of "word problems". This is not to say that the semantic characteristics can be divorced from the logical structure of the context. They probably cannot be. However, Carpenter & Moser (1982) were concerned to identify the type of action or relationship which is represented in most addition and subtraction contexts which are simple enough for primary-aged children to handle. This researcher modelled her selection of contexts on the Carpenter & Moser taxonomy (which will be discussed later), and although she did not analyse their semantic characteristics, such an analysis might possibly be worthy of consideration by other researchers.

THE FINDINGS

The findings will be described and discussed in terms of the questions posed earlier.

> 1a IN EXECUTING ADDITION AND SUBTRACTION OPERATIONS, WHAT USE IS MADE OF FINGERS, CUBES AND TALLY MARKS AND IS THIS 'SUSPENDED REALITY' OF COUNTING AIDS EASILY TRANSLATABLE TO AND FROM A REALISTIC, EVERYDAY ADDITION OR SUBTRACTION CONTEXT?

In teaching the children to formally add and subtract using the method of Bidirectional Translation, addition was taught first, and then subtraction. Some teachers argue for teaching the skills 'together' in the name of trying to establish the complementary nature of the skills. However, Fehr & Phillips (1967) argue that:

> it is not good practice to introduce addition and subtraction at the same time. They are at first two distinct and different operations. After the addition concept has been thoroughly developed, we can develop a new concept called subtraction. When the latter concept is well understood, then we can relate the two operations.

In the early part of the protocol for addition (steps 1, 2 and 3) some subjects counted the cubes, others counted correspondingly on their fingers, while others still, used their knowledge of number facts and made no reference to fingers or cubes. That some subjects were observed counting on their fingers when the sub sets to be totalled were already modelled in front of them in the form of Unifix Cubes, the researcher found surprising. On the face of it, this seemed a senseless performance: the countables were present in concrete form so why model further on the fingers? On reflection, and after subsequent discussion with those subjects on the use of fingers for counting, however, the strategy is both reasonable, and helpful to the children themselves. It is reasonable from the point of view that we always have fingers available in the way that we do not always have other concrete aids such as Unifix Cubes. More interesting to the researcher, however, were the subjects' comments when they were questioned about using their fingers for totals greater than ten, as say, in five plus seven. The responses were of three main types. An example of each is listed below.

> Type 1 You can use bits of your body to help, like your eyes; or you can use your elbows.

This certainly seems to reflect ingenuity in that the subjects were not prepared to become 'stuck' or be put off.

> Type 2 Well then, I've got five fingers on this hand so that's five. On the other hand I've got five fingers and I'll put two magic fingers on to make the seven.

This second type of response seems more sophisticated than the first. Here seems to be the beginning of mental representation without concretization. The subject does not have to see the two objects (or models of them) to accept that they are there for the purpose of being included in the addition operation. It is almost as if the subject can 'concretize in the mind' (if one can forgive the contradiction in terms!).

> Type 3 I don't use my fingers for all of the numbers. I just say, "There's five", and then I count out seven fingers. Then I say, "five, six, seven, eight, nine, ten, eleven, twelve".

This third type of response seems the most sophisticated of the three. In essence the subject is demonstrating the transition to the 'counting on' strategy which is said to evolve out of the 'counting all' strategy (Carpenter & Moser, 1982; Fuson, 1982).

By way of contrast, subjects did not initially model subtraction operations on their fingers. They were quite prepared to use the Unifix Cubes provided. However, on observing this, and remembering how some of the same subjects had spontaneously used fingers to model addition, the researcher at one point early in the protocol for subtraction, removed the cubes, leaving the subjects with only their own resources. Those subjects who had a repertoire of number facts appeared to make use of them in that there was no overt indication of counting behaviour. All subjects were, however, encouraged to model the subtraction operation on their fingers: displaying the minuend, folding down the subtrahend and counting the number of fingers still displayed. The use of fingers for subtracting was quickly and easily assimilated by the subjects.

Hughes (1986) would seem to be right when he says that we should show children "how to use their fingers more effectively" when carrying out number operations.

The subjects enjoyed the Let's Pretend features (step 2) of having the Unifix Cubes represent objects in the real world. Their laughter was evidence of this. If the cubes were serving as sweets some of the subjects would pretend to eat them, or if they were cars the subjects would provide accompanying sound effects. This again suggests that we should not underestimate the importance of modelling procedures explicitly.

Now and again, some of the subjects made unprovoked (by the researcher) statements such as "this is just like sums" or "we're doing real sums now" when they were required to translate verbal contexts into arithmetical notation (step 4). These remarks suggest that the subjects did have an awareness of arithmetical operations. Whether this was because these subjects had seen others 'doing sums' or were perhaps themselves 'doing sums' at home, is not clear, and was not pursued by the researcher.

Many of the subjects perceived the parallels between the addition and subtraction protocols because, frequently, they made comments such as, "we've done this before" and, "this is just like adding only we're taking away". Very informally, the researcher reacted to these comments by asking the subjects how they could reverse the effects of the subtraction operation and they were able to suggest that an addition operation was needed.

There was a lot of enthusiasm to provide 'silly stories' (step 5). This was clearly a meaningful activity. The subjects frequently remarked on their enjoyment of it, as they clamoured to contribute yet another verbal context for a numerical representation. The motivation, for the children, appeared to the researcher to be almost totally intrinsic. Granted, it would probably be reinforcing to subjects for them to have their contributions appreciated by peers but if peer approval alone was what was motivating the children, then they could just as easily have taken recourse to the most facetious contexts imaginable. But an alternative explanation is that the subjects did really want to rehearse their skill in translating to a verbal context; in the sense of it being a Piagetian schema. This translating-to-a-verbal-context schema is then the structure within which the child 'checks out' that he/she is making sense of the formalism of, in this study, addition and subtraction.

The pictorial representation of the subtraction protocol was, perhaps, the most problematic part of the protocol for most of the subjects. As anticipated by the researcher, the subjects initially drew pictures to represent the minuend <u>and</u> the subtrahend, and then proceeded to count the total number of pictures drawn. However, in that the subjects had made their own difficulty explicit, the researcher was able to ask them to 'think again'. When the subjects tried to provide a verbal context for their drawings they found they were contradicting themselves, making an initial quantity smaller and yet ending up with more than they had started with!. By stressing to the subjects that they had to show that something had happened to the initial quantity but that the initial quantity had to be available for all to see, the subjects eventually resolved their problem by drawing the minuend and crossing out the subtrahend.

The use of tally marks was quickly and easily assimilated by the subjects. This should not be surprising, given what has earlier been said about tally marks. In making their choice, about what type, if any, of counting aid to use, there seemed to be no pattern. Most subjects used all three types at different times and their choice seemed to depend as much as anything on what they felt like at the time; and this seems quite reasonable. <u>If</u> for us as adults, getting to work were equally practicable in all respects whether we travelled by bicycle, car or train, no doubt our choice would also depend on what we felt like at the time!

In summarizing an answer to the first question it can be said that subjects made considerable use of counting aids which they fully recognized as being a means to an end.

1b CAN THE VERBAL CONTEXTS SUPPLIED BY THE SUBJECTS BE CATEGORIZED ACCORDING TO SOME CRITERION?

In the verbal contexts provided by the subjects for steps 6, 7 and 9 in addition, the hierarchies of classification were very crude. Of the sixteen subjects, only seven consistently used immediately recognizable classifications such as boys, girls, toys, animals, sweets. Of the remaining nine subjects, the reader/observer would be strained to detect the classifications; which were mostly things to eat (for example carrots and lollipops) and things in the street (for example buses and houses). Other sub sets in the verbal contexts, such as people and Mars Bars, or kittens and ice-cream defy classification in all but the crudest sense of 'things'. It is perhaps significant that the seven subjects who did use easily recognizable classifications were the group of children earlier considered by the researcher to be the most proficient in terms of counting skill. These were also the subjects who had a repertoire of number facts at their disposal and were less reliant on counting as a means of obtaining an answer to an addition operation. But since, as Gelman & Gallistel (1978) point out, children do not necessarily restrict themselves to counting collections of identical objects, it is perhaps reasonable to expect those subjects who computed addition operations by counting (as distinct from using number facts) would also provide verbal contexts in which miscellaneous and incongruent sub sets were to be added.

Perhaps as a direct consequence of the nature of the subtraction operation, the hierarchy of classification appeared to be more refined than it had been for addition. What was subtracted and what was left were always sub sets of the original class of objects. In effect, the subjects were constrained by the subtraction operation and could do little else in respect of the classification content.

The above analysis is, however, too simplistic. It implies that verbal contexts for all addition and subtraction operations are essentially homogeneous in nature and that while syntax, lexical items and context may vary, the underlying logical structure of the context is the same for all. Such an assumption is questionable.

Carpenter & Moser (1982) suggest three dimensions on which the structure of verbal contexts can be analysed.

Firstly, there is the ACTIVE-STATIC dimension. Active contexts involve explicit or implicit reference to action that takes place. In other words something has to be 'done', usually to the initial quantity. Static contexts, on the other hand, require no action but rather a 'contemplation' or 'consideration' of two given quantities.

Secondly, there is the SET INCLUSION-DISJOINT QUANTITIES dimension. Set inclusion contexts are those where two of the quantities are necessarily a sub set of the third. In other words, either the two given quantities add up to the third, unknown quantity or the unknown quantity is what is left after a known quantity has been subtracted from a second, larger, known quantity. Disjoint quantities contexts imply no set-subset hierarchy, however. A disjoint quantities context can, then, involve 'things' which would not necessarily go together in a conventional, adult way because the disjoint quantities implies comparison.

Finally, there is the INCREASE-DECREASE dimension. This dimension only applies to contexts which already fulfil the action criterion in the active-static dimension, above. Contexts which involve action will result in an increase or decrease of the initial quantity.

The permutations arising from combining these dimensions are,according to Carpenter & Moser (1982), as follows:

1. <u>The Joining Class</u> ACTIVE, SET INCLUSION, INCREASE;
2. <u>The Separating Class</u> ACTIVE, SET INCLUSION, DECREASE;
3. <u>The Part-Part-Whole Class</u> STATIC, SET INCLUSION;

4. The Comparison Class STATIC, DISJOINT
5. The Equalizing Add-On Class ACTIVE, DISJOINT, INCREASE
6. The Equalizing Take-Away Class ACTIVE, DISJOINT, DECREASE

A further complication to this analysis of verbal contexts is that for each of the different classes of verbal contexts there can be both addition and subtraction operations. The following outline may help to make clear what Carpenter & Moser (1982) seem to be saying.

The Joining Class of verbal context has an initial quantity and some direct or implied action that causes a change in the quantity. The action causes an increase in the quantity. An addition operation in this class might be: Sheena has 3 pencils. Bob gave her 4 more pencils. How many pencils does Sheena have now? A subtraction operation in this class might be: Sheena has 3 pencils. How many more does she have to buy to have 7 pencils altogether?

The Separating Class of verbal context has an initial quantity and some direct or implied action that causes a change in the quantity. The action causes a decrease in the quantity. An addition operation in this class might be: Fred had some sweets. He gave 2 to Linda and now has 4 left. How many sweets did Fred have to start with? A subtraction operation in this class might be: Fred had 6 sweets. He gave 2 to Linda. How many sweets has Fred got left?

The Part-Part-Whole Class of verbal context describes a static relationship between two distinct quantities that are parts of a whole. An addition operation in this class might be: There are 4 hats and 2 coats in the cupboard. How many garments are in the cupboard? A subtraction operation in this class might be: Mary has 9 flowers. 6 of them are red and the rest of them are blue. How many blue flowers has Mary got?

The Comparison Class of verbal context again describes a static relationship between two quantities but this time the quantities need not be parts of a whole. An addition operation in this class might be: Jimmy has 2 more footballs than Susie has dolls. Susie has 3 dolls. How many footballs has Jimmy got? A subtraction operation in this class might be: There are 5 footballs and 3 dolls in the playroom. How many more footballs are there?

Both of the Equalizing Classes of verbal context are complex to read about, and for most practical purposes quite unrealistic. However, a simplified interpretation would be that they firstly involve comparing two quantities and secondly that one of the quantities has to be changed so that the two quantities become equal. An addition operation in the Equalizing Class might be: There were 6 boys in the football team. 2 more boys joined the team. Now there are the same number of boys and girls in the team. How many girls are in the football team? A subtraction operation in this class might be: There are 5 paint brushes and 3 pots of paint on the table. How many paint brushes do I need to take away so that there will be the same number of paint brushes and pots of paint on the table?

Carpenter & Moser (1982) claim that their taxonomy of verbal contexts has validity in that in their empirical work "children's solution processes clearly reflect" Carpenter & Moser's distinctions between types of verbal contexts. Using Carpenter & Moser's taxonomy as a criterion, the question turns on the extent to which the verbal contexts offered by subjects in this study can be classified.

For each of the 16 subjects, ten verbal contexts were noted during the course of the teaching protocols, five contexts for addition and five for subtraction. The results of analysis of the verbal contexts offered by subjects were quite clear cut. All of the subjects when providing a verbal context for an addition operation, offered a context which fits into the Part-Part-Whole Class and all of the subjects when providing a verbal context for a subtraction operation, offered a context which fits into the Separating Class. Some examples are listed below.

<u>Figure 7.2 Examples of Verbal Context Offered by Subjects for addition</u>

1. I bought 2 ice-creams and 3 worms. Altogether that made 5 things.
2. I went to the sweetie shop and I got 3 ice-creams and 3

lollipops and altogether that made 6 things.

3. I saw 2 houses and 4 skyscrapers and that made 6.
4. I was walking down the street. I saw 4 girls and 2 boys. Altogether I saw 6 people.
5. I went into a shop and bought 1 man and 5 apples. That made 6 things.

<u>for subtraction</u>

1. I got 4 balls and sold 1 of them. That left 3.
2. I saw 3 cars and they all drove away. That left 0 cars.
3. There were 5 houses in the street. 4 went on fire and that left 1 house.
4. I went to the toy shop and bought 3 dolls. 1 of them got broken and my mum chucked it in the bin. That left me with 2 dolls.
5. I had 6 precious things and the robber stole 1 of them and that left 5.

Whether or not the children can deal with other types of verbal context, as defined by the Carpenter & Moser classification, remains to be seen. All that is being said at the moment is that the verbal contexts offered by subjects in this study fall into two main groups - that of Part-Part-Whole for addition and that of Separating for subtraction.

1c DOES THE USE OF OPERATOR SIGNS EASILY BECOME INCORPORATED INTO THE NUMERICAL REPRESENTATION?

When the subjects were required to draw a 'silly story' in the addition protocol (see steps 6 and 7) they, of their own volition, supplied the 'plus' and the 'equals' sign. This is in sharp contrast to Hughes's (1986) findings where the children in his study made no spontaneous use of the operator signs. Moreover, in this study, the subjects also provided the total numerosity without being instructed to do so. It would seem reasonable to explain this phenomenon in terms of the preparatory steps to this point in the protocol having become so routinized that the subjects predicted that they would be required to 'complete' the story. But non of them completed the story by drawing the total numerosity. Each and all of them supplied the missing parts in symbolic notation. Perhaps the subjects were beginning to realize the utility of formalization. On the other hand, perhaps this was simply a chance finding and a function of the subjects being taught in a group situation where they were allowed to (and encouraged to) talk about what they were doing.

It will be recalled that subjects found it much more difficult to pictorially represent subtraction. When they eventually found a means of depicting the minuend and the subtrahend, they made no effort (as they had previously done in addition) to supply the operator signs or the number which represented the outcome of the operation. The subjects were, however, perfectly willing to comply with the researcher's suggestion that the complete 'number story' should be recorded underneath the pictorial representations for subtraction and appeared to have no difficulty in doing so.

Because of Hughes's (1986) findings that young children tend to disregard operator signs, it was anticipated, in this study, that a similar phenomenon might present itself. However, this fear proved unfounded. Perhaps because in the protocols, time had been expended on explaining the significance of operator signs, the subjects did not regard them as arbitrary marks but as meaningful symbols which were as important to the 'number story' as the numbers themselves. Certainly there was no reluctance on the part of the subjects to use operator signs when they were undertaking the conventional addition and subtraction exercises.

2a CAN THE CHILDREN TRANSLATE FROM A VERBAL CONTEXT TO OBTAIN THE SOLUTION TO AN ADDITION OR SUBTRACTION OPERATION? (This is subsequently referred to as the utilizability of the concept).

Within each of the protocols, translating from a verbal context provided by the teacher could not distinguish between having the concept of addition and having the concept of subtraction. Bearing in mind that addition was taught first and then subtraction, the test of 'concept acquisition' had to wait until after subtraction had been taught so that the children could by process of elimination, or use of negative example, decide that subtraction was not addition and, conversely, that addition was not subtraction.

The usual means of testing for 'concept acquisition' is to explore whether or not the learner can apply or utilize the concept. This was duly done using the procedure outlined earlier.

Table 7.3 The Number of Correct Responses for Utilizability of Addition and Subtraction

Ss	Addition	Subtraction	Total (out of 10)
1	5	5	10
2	5	5	10
3	5	5	10
4	3	4	7
5	5	5	10
6	5	5	10
7	5	5	10
8	5	5	10
9	4	5	9
10	5	5	10
11	5	5	10
12	5	5	10
13	5	5	10
14	5	5	10
15	4	3	7
16	5	4	9

Credit was given if it was obvious that the subject was appropriately increasing or decreasing a quantity. In other words a slight computational error of margin (+ or - 1 of the correct answer) is included in the scores. Inspection of Table 7.3 shows that most of the subjects were well able to distinguish between addition and subtraction. The high incidence of correct responses makes statistical analysis irrelevant. The simple answer to the question posed here is that subjects could utilize addition and subtraction concepts within the number domain of 10.

2b CAN THE CHILDREN TRANSLATE FROM A VERBAL CONTEXT TO IDENTIFY AN ADDITION OR SUBTRACTION OPERATION? (This is subsequently referred to as the evocability of the concept).

Table 7.4 The Number of Correct Responses for Evocability of Addition and Subtraction

Ss	Addition	Subtraction	Total (out of 10)
1	5	5	10
2	5	3	8
3	4	4	8
4	4	4	8
5	4	3	7
6	4	4	8
7	5	5	10
8	4	3	7
9	3	4	7
10	4	2	6
11	4	2	6
12	4	4	8
13	4	3	7
14	4	4	8
15	4	3	7
16	4	3	7

Inspection of Table 7.4 shows that subjects less readily identified a subtraction operation as being appropriate than they did an addition operation. Only one subject performed better on subtraction than on addition while eight subjects performed better on addition than on subtraction, with the remaining seven subjects performing equally well on addition and subtraction. On a related t-test, considered to be very robust, a significant difference was found between performance on addition and subtraction, $t=2.8248$ ($p<.02$). The null hypothesis was that differences in performance in evoking addition and evoking subtraction were due to guessing. Because the observed value of t was larger than the critical value, the null hypothesis can be rejected. The difference in scores between evoking addition and evoking subtraction would then seem to suggest that evoking subtraction was more difficult for subjects than evoking addition.

2c IS THERE ANY APPARENT RELATIONSHIP BETWEEN BEING ABLE TO UTILIZE AND BEING ABLE TO EVOKE ADDITION OR SUBTRACTION CONCEPTS?

Table 7.5 The Number of Correct Responses for the Utilizability and Evocability of Addition

Ss	Utilizability	Evocability	Total (out of 10)
1	5	5	10
2	5	5	10
3	5	4	9
4	3	4	7
5	5	4	9
6	5	4	9
7	5	5	10
8	5	4	9
9	4	3	7
10	5	4	9
11	5	4	9
12	5	4	9
13	5	4	9
14	5	4	9
15	4	4	8
16	5	4	9

Inspection of Table 7.5 shows that only one subject scored better on the evocability than on the utilizability of addition, four subjects scored equally well on the evocability and utilizability of addition but eleven out of the sixteen subjects scored better on the utilizability of addition than they did on the evocability of addition.

Table 7.6 The Number of Correct Responses for the Utilizability and Evocability of Subtraction

Ss	Utilizability	Evocability	Total (out of 10)
1	5	5	10
2	5	3	8
3	5	4	9
4	4	4	8
5	5	3	8
6	5	4	9
7	5	5	10
8	5	3	8
9	5	4	9
10	5	2	7
11	5	2	7
12	5	4	9
13	5	3	8
14	5	4	9
15	3	3	6
16	4	3	7

Inspection of Table 7.6 shows that twelve of the subjects scored better on the utilizability than on the evocability of subtraction while the remaining four subjects scored equally well on the utilizability and evocability of subtraction.

On the criterion of scoring at least four out of five test items correct, all but three of the subjects (a different one in each condition) were able to utilize addition, to utilize subtraction and to evoke addition; but only half of the subjects were able to evoke subtraction. The breakdown of figures looks like this:

Table 7.7 Observed Frequencies of Evidence of Conceptualization

Scoring	Util(+)	Util(-)	Evoc(+)	Evoc(-)
4/5	15 subjs.	15 subjs.	15 subjs.	8 subjs.
< 4/5	1 subj.	1 subj.	1 subj.	8 subjs.

With the observed frequencies in three of the categories being so low, the expected frequencies, in these instances, fall below 5. Thus the use of the Chi-square test would yield unstable results. Statistical testing is, therefore, inappropriate at this point. There appears to be a relationship between being able to evoke and utilize addition. However, this apparent relationship does not seem to hold for subtraction.

3a CAN THE CHILDREN, IF REQUIRED TO DO SO, ARTICULATE HOW THEY OBTAINED A SOLUTION WHEN THEY HAD TO UTILIZE A CONCEPT IN TEST PERFORMANCE?

It had been a declared aim of the methodology of Bidirectional Translation, that the learners be encouraged to verbalize what they were doing: explaining how they did, and speculating on how they could, effect a computation. It seemed apposite to explore how supportive this strategy was to the subjects, particularly in the light of the uncertainties revealed in the relationships between the evocability and utilizability of addition and subtraction.

It will perhaps facilitate discussion to be able to refer to subjects' actual scores. The table below shows the distribution of correct responses for each of the classes of verbal context when the subjects were evoking and utilizing addition (+) and subtraction (-).

Table 7.8 Summary of Correct Responses in each Class of Verbal Context for each type of Conceptualization

	JOIN.	SEPA.	PART.	COMP.	EQUA.	total
EVOC(+)	16	3	16	15	16	66 out of 80
EVOC(-)	2	16	16	4	14	52 out of 80
UTIL(+)	16	14	16	14	16	76 out of 80
UTIL(-)	15	16	16	13	16	76 out of 80
total	49	49	64	46	62	270 out of 320

As can be seen from Table 7.8, there was no particular example of verbal stimulus which was beyond the abilities of every subject. But Joining (Evoc -), Separating (Evoc +) and Comparison (Evoc -) 'polled badly' relative to the other examples. In the Joining Class only two subjects could state that a subtraction operation was required, yet fifteen subjects in a parallel condition could compute the subtraction operation. In the Separating Class only three subjects could state that an addition operation was required, yet fourteen subjects in a parallel condition could compute the addition operation. In the Comparison Class only four subjects could state that a subtraction operation was required, yet thirteen subjects in a parallel condition could compute the subtraction operation.

Somewhat concerned about this discrepancy, the researcher generated a series of isomorphs of those verbal contexts in which the subjects had performed so badly. She provided the subjects with Unifix Cubes and asked them to demonstrate with the cubes their processes for finding the answer. She further asked each subject, on completion of the task, whether an addition or subtraction operation had been performed. The results of this follow up study were consistent with the original findings and, as such, need not be discussed at length. Observations of the subjects' behaviours are, however, worthy of recording.

In the Joining/Subtraction context the subjects set out the initial quantity of cubes as an addend, counted on the appropriate number of cubes till they reached the requisite number and then counted to what amounted to the second addend. When asked if they were adding or subtracting, the subjects stated that they were adding because they were taking more cubes from their 'bank'.

In the Separating/Addition context the subjects constructed the addends and summed them but still maintained that they were subtracting because part of the quantity had been 'given away' and only some of it was 'left'.

In the Comparison/Subtraction context the two disjoint quantities were constructed and the difference observed, and counted. Again the subjects maintained they were adding because, they argued, the smaller of the two disjoint quantities together with the difference was equal to the larger of the disjoint quantities.

Clearly then, the subjects could give reasons for their behaviour and, what is more, their explanations shed possible light on the earlier, observed 'failure' of half of the subjects to evoke subtraction.

Firstly, the subjects had tactical strategies for dealing with the verbal contexts. These strategies would appear to be based on counting. If the subjects do not have number facts available for instant recall (and there was very little evidence that they had) their own well developed counting skills are the only meaningful strategies they possess for addition and subtraction. In the case of Joining and Comparison contexts for subtraction it made sense to perform the subtraction operation by complementary addition or 'adding on'.

Secondly, the subjects have, as yet, poorly developed executive strategies. They identify key words such as 'more' and 'altogether' with addition, and 'left' and 'take away' with subtraction and will assign verbal contexts containing these words accordingly. This does not mean, at this early stage of their formal education, that the children act on their executive decision. The Separating context for addition is a case in point, where the subjects counted additively but still claimed that it was a subtraction operation.

That the subjects could 'talk through' what they were doing is not in doubt, when they were utilizing the concepts. Their evocability of concepts was, however, on their own terms. They viewed complementary addition as addition (what could be more reasonable!) and they also viewed the comparison of disjoint quantities as addition. These 'misconceptions' as to what is classed as addition and what is classed as subtraction do, however, appear to have affected the subjects' performance on the evocability of subtraction. This point will be taken up later.

3b GIVEN A PREDETERMINED SELECTION OF CLASSES OF VERBAL CONTEXT WITH WHICH TO TEST CHILDREN'S CONCEPTUALIZATIONS, IS THERE ANY APPARENT RELATIONSHIP BETWEEN TEST PERFORMANCE IN TRANSLATING FROM A CONTEXT AND EARLIER PRE-TEST PERFORMANCE TRANSLATING TO A CONTEXT?

Inspection of Table 7.8 shows that, with the exceptions discussed in the previous section, children were well able to translate from verbal contexts.

It will also be recalled that the researcher's classification of verbal contexts provided by subjects during the teaching protocol, using the Carpenter & Moser (1982) taxonomy, resulted in subjects' offerings being exclusively Part-Part-Whole for addition and Separating for subtraction.

Given that Part-Part-Whole and Separating Classes were favoured by the subjects initially, it is not surprising that in the test performance both categories resulted in high performance scores for both evocability and utilizability. But given also, the restricted classification of texts offered by the subjects, it is interesting that they performed so well on contexts belonging to other classifications.

Allowing for the possible 'misunderstandings' on the part of the subjects in evoking operations in the Joining Class for subtraction, the Separating Class for addition and the Comparison Class for subtraction (as discussed earlier) it seems reasonable to conclude that the subjects' experience of providing verbal contexts in the exposure to Bidirectional Translation possibly enabled the subjects to cope with a range of verbal contexts.

3c WHAT CLAIMS CAN BE MADE FOR THE METHODOLOGY?

Claims for the methodology will be discussed in terms of the conceptualizations evidenced, and will not include considerations such as children's enjoyment of/engagement in the protocol or mechanical representations which, though important in themselves, are not the main focus of this study.

The subjects were clearly able to utilize addition and to utilize subtraction. This suggests that subjects could distinguish between addition and subtraction operations. That they could extract the appropriate operation from the verbal context demonstrates that they had conceptual understanding of addition and subtraction. They could, in essence, perform successfully on the translation problem. Given what was earlier claimed for, in the synonymity of successful performance on the translation problem and conceptual understanding, it can be argued that the subjects had cognitive control of both addition and subtraction (albeit within a limited number domain). Crudely put, the subjects understood what they were doing when they added or subtracted, in that they knew <u>when</u> to add or subtract and <u>how</u> to add or subtract.

In the light of this finding therefore, it would seem reasonable that young children should be exposed to the formalism of addition and subtraction and, presumably, to other types of arithmetical operations, by the type of approach adopted in Bidirectional Translation. If children were able, in much higher proportions than are currently documented, to perform successfully on the translation problem, then much of the concern which has prompted this whole piece of research would die away.

But as Skemp (1971), Donaldson (1976) and Flavell (1979) have pointed out, conceptual understanding is not restricted to only having cognitive control of the concepts in question. Mature conceptualization involves also having metacognitive control. Translated into the terms of this study, this means that being able to add and subtract effectively (yes, even when the operation is couched in a verbal context) is not enough. The subjects must, further, know that they could add or that they could subtract (without necessarily doing so) to effect an appropriate solution to a translation problem. In other words the subjects must be aware of their own mental activity insofar as they can determine in advance of actual implementation, or hypothesize in the absence of actual implementation, which operation is needed for translation problem solution.

Now it could be argued that the researcher's pursuit of metacognition in primary one children is somewhat ambitious given the research evidence (Piaget, 1928; Reid, 1966; Karabenick & Miller, 1977; Renwick, 1984; Garofalo & Lester, 1985) which suggests that children of 5 - 6 years of age are incapable of exercising metacognitive control. However, given the views of Brown (1978), Flavell (1979), Brown & DeLoache (1983) and Nisbet & Shucksmith (1984, 1986) that metacognition is an influence on cognition and that at least some notion of metacognitive control can begin to be developed in young children, it is the researcher's considered opinion that metacognition through the medium of evocability is worthy of exploration at least.

At first glance, the subjects appeared not to be able to evoke the concepts as readily as they could utilize them. However, as further exploration found out, the subjects' 'misconceptualizations' of some types of addition and subtraction would according to the subjects' own reasoning, appear to be tied to the semantics of the context which 'carries' the logical structure of the operation. There have been many investigations into the analyses of verbal contexts (Carpenter & Moser, 1982; Nesher, 1982; Verngaud, 1982; Dickson et al, 1984) and of how differing combinations of syntax and vocabulary variably affect performance. Engagement in this debate is not part of this study. However, that the same or similar words can be arranged in a variety of legitimate combinations to convey different meanings cannot be ignored since, as has been demonstrated in this study, different types of verbal context will affect the subjects' apprehension, not of what they are necessarily doing but of what they think they are doing; the case in point being when subjects believed themselves to be adding when they were subtracting and conversely believed themselves to be subtracting when they were adding. But the subjects' 'errors' were only errors on the criterion of mature, adult, mathematical conceptualization. There was nothing, actually, inherently <u>wrong</u> in what the subjects were doing or in their thinking about what they were doing. Without the protocol of Bidirectional Translation to establish amongst the children that discussion about what one was actually doing was the 'norm', it seems unlikely that any of the insights gained in this study would have been available to the researcher.

This difference between what the adult accepts as so, and what the child believes to be so, has implications for the teaching of addition and subtraction. The child needs help to adjust <u>his/her</u> conceptualizations to those commonly held in the mathematical world; just in the way that the child has to adjust from using his/her own idiosyncratic list of counting words to the conventional listing. Without the common frames of reference it is impossible to communicate clearly with others. The child who never learns that complementary addition is subtraction will have extreme difficulty in discussion with the teacher who does not see that complementary addition can be viewed as addition, particularly in the later stages of formal education when addition and subtraction are but a part of the task in hand. But how is the teacher to know that the child conceptualizes complementary addition as addition and not as subtraction if there has been no effort made to elicit what the child thinks he or she is doing?

For the young child at an early stage in number work, the strategy of latching on to key words such as 'more' or 'left' to make an executive decision does not seem to be too dangerous. His/her heavy reliance on the tactical business of counting ensures that a reasonably accurate computation can be performed. Up to a point this seems perfectly satisfactory. But there frequently comes the time (most noticeably in the middle and upper primary stages) when the teacher, in helping the child 'digest' the translation problem, says something like, "tell me how you are going to find the answer" (Dickson et al, 1984) or "what shall we do?" (Kilpatrick, 1981), particularly as by then the translation problem may be a complex one in the sense of several operations to be computed (Charles & Lester, 1984). What is then counter productive is to allow the child to continue to believe that all verbal contexts can be represented numerically simply by applying the operations suggested by the key word. In the context, 'Fiona had some carrots. She gave 3 to the rabbit and now has 2 left. How many did she have to start with?' the word, 'left' is <u>not</u> a cue to subtract. Similarly, in the context, 'Mr Brown has 4 shirts. How many more shirts does he need to have 6 altogether?' the application of the addition operation will result in the wrong answer since the word, 'more' is not the cue to add. However, if the child has been successful with this approach (because it has always previously happened that 'more' and 'left' have correctly been associated with addition and subtraction operations respectively) and believes the he/she always will be successful with this approach, then the child is likely to use it even when it is inappropriate. One possible corollary from this is that the child abandons his/her earlier, intuitive attempts to understand the relationship expressed in the verbal context and instead adopts some quasi-mathematical rote heuristic.

The young subjects in this study certainly seemed to appreciate the need to search for meaning in the verbal context. As teachers we must nurture this phenomenon and not allow it (through an overdue emphasis on 'performance') to become subjugated to correct answers or speedy progression through a prescribed syllabus. In turn this suggests two possible recommendations to be included in policies for the teaching of mathematics:

firstly, that there be an acknowledgement of the necessity to 'revisit' abstract mathematical concepts over time. Only then can what were initially fragmented concepts have a chance of becoming developed in anything resembling a 'complete way';

secondly, that there be a genuine attempt on the part of the teacher to elicit from the child exactly what the child thinks he/she is doing when working on a translation problem. Just in the way that no teacher would wittingly allow a child's incorrect overt performance to continue without some attempt to help the child make the correction, so too does the teacher need to appreciate that faulty performance may be due not to a lack of competence but to a less mature or disoriented conceptualization of the task. One way of accessing this conceptualization is through some exploration of evocability. Granted, the measure of evocability used in this study may be deemed to be a bit crude, but it is a start.

In summary:

(i) the subjects appeared to respond well to being taught the formalisms of addition and subtraction through the method of Bidirectional Translation;

(ii) in their operations the subjects could clearly distinguish between addition and subtraction (that is they could utilize addition and subtraction) which is tantamount to saying that the subjects had cognitive control of the two concepts;

(iii) in their reflections on hypothetical operations there appeared to be some confusion as to what constitutes addition and what constitutes subtraction (that is there was some difficulty in correctly evoking addition and subtraction) which in turn suggests that the subjects' metacognitive control of the concepts was less refined;

(iv) this confusion is understandable insofar as it may be a function of the type of verbal context used, but nevertheless is a confusion which should be regarded as legitimate and as worthy of 'teaching' as any aspect of pedagogical content, if the different types of verbal context are accorded equal veracity.

THE EXTENSION OF THE EMPIRICAL WORK:

THE EXPERIMENT

The findings of the Main Study were pleasantly rewarding to the researcher. It did seem possible to teach addition and subtraction in some comprehensible fashion such that learners were aware of what they were doing. They knew when and how to compute. They could make sense of their computations in terms of hypothetical, real-life scenarios. And they could speculate on whether addition or subtraction operations were an appropriate 'fit' for the particular translation problems under consideration. There was still, however, a large unanswered question, "Was Bidirectional Translation's 'success' merely a function of the idiosyncrasies of this particular teacher?". Put more prosaically (or scientifically?) were the findings of the Main Study conducted with a tiny sample, generalizable to a larger population? To find, or try to find, an answer to this question, the methodology had to be tested in some systematic way and, as characterizes much educational research, the most appropriate vehicle for such testing was seen as the experiment, a means of controlled observation.

Factors to be considered

While the experimental style of data collection does allow conclusions to be drawn about cause and effect, it is nevertheless an approach which is fraught with difficulties.

These difficulties can be reduced to what Campbell & Stanley (1963) refer to as two principles: those of internal and external validity.

Internal validity means that the results obtained in the experiment are due to the experimental 'treatment' rather than to uncontrolled, extraneous factors. In terms of this study, performance on addition and subtraction test items would have to be demonstrably tied to the teaching methodology.

External validity means that the results obtained in the experiment would apply in the real world, at other times, to other groups of people, in other geographical locations. In terms of this study, could any qualified primary teacher use the methodology?

The principles of internal and external validity are inextricably linked. The tighter the control of the experiment itself (in terms of controlling variables) the higher the probability that the study is internally valid. However, the elimination of uncontrolled variables which freely reign in the real world proportionately reduces the external validity of the study. And yet, external validity is of little value if one can have no confidence in the the internal validity of the experiment. The solution lies in achieving the best possible trade off between factors involved in external validity and those involved in internal validity. But what are those factors? Campbell & Stanley (1963) refer to the following:

for internal validity

1. the history of the subjects: This refers to 'events' in the subjects' environment, which are beyond the control of the researcher, and which may have a favourable or disturbing effect upon the performance of the subjects such that performance measures were being wrongly attributed to the experimental 'treatment'. Limitations on internal validity by virtue of history are dealt with by using a control group which can be expected to have experienced the same 'events' (that is, can be expected to have the same history) as the experimental group.

2. the maturation of the subjects: This refers to maturational and developmental experiences which occur normally and which could, rather than the experimental 'treatment', be responsible for a particular outcome. Particularly over extended time, it can be difficult to determine whether improved performance is due to the independent variable, or to maturation, or to an interaction between the two. Here again, the use of a control group composed of persons who can be expected to have had the same or similar developmental experiences can enable the researcher to control for this confounding effect.

3. testing: Tests may make subjects more aware of hidden purposes of the researcher, and as a result may act as a

stimulus to change on the part of the subjects. This is particularly problematic, it seems, if there is pre-testing prior to the experimental 'treatment'. The subsequent post-test may not then be measuring the effects of the independent variable but be measuring the consequent effects of the pre-test experience. In traditional experimental designs, this bias can be avoided by not having a pre-test.

4. <u>instrumentation</u>: The measuring tools or techniques must be as reliable and as up-to-date as possible, otherwise the experiment's validity is threatened. Additionally, however, the researcher must handle his/her measuring instruments with care. As the experiment proceeds, the researcher can, unwittingly, affect the measures obtained by giving different subjects different cues, by asking different subjects different questions, or by coding the data differently. To avoid unreliable and invalid information, it is important to have at least one but preferably both of the following:

to have a predetermined format for asking questions/recording data;

to have interviewers/observers constant across time.

5. <u>selection of subjects</u>: The selection of subjects is probably the most important factor in experimental research. To evaluate properly the influence of the 'treatment' variable it is necessary that both the experimental and control groups be as closely as possible equivalent in respect of all the factors that may influence the dependent variable except for the

factor(s) chosen as the independent variable. In practice it is extremely difficult to select the two groups from the population who are equivalent or comparable in all respects and the usually favoured resolution to this difficulty is to use some form of randomization: either selecting subjects randomly, or randomly assigning subjects to different treatments, or randomly assigning experimental 'treatments' to selected groups.

6. stability: Test findings can be unreliable in that they can occur once but not consistently. To control for this factor the data need to be examined statistically. Statistical tests indicate the probability of the findings being due to chance.

for external validity

1. The Hawthorne Effect: This refers to the possibility that participants in an experiment, just because they know they are participants, will react more positively towards the independent variable than the independent variable really justifies, and thereby enhance the findings. The phenomenon of the Hawthorne effect often operates in experimentation in curriculum research: the subjects, pleased at having been singled out to participate in an experimental project, react more strongly to their pleasure than to the 'treatment'. (But when such projects are tried on a non-experimental basis they often yield different results.) This means that performance measures may be more a function of the researcher's

intervention per se than the specifics of the intervention. This reactive effect can contribute considerably to the invalidity of the findings and there is no way of eradicating it. In order to make some judgement as to the 'true' effects of the experimental treatment, it is important to cause the control subjects to feel that they too are participating in the experiment by the researcher's introduction of some pseudo intervention which makes the control subjects feel they are involved and are important but which has no relation to the independent variable being evaluated. In this way the researcher can attempt to 'create' parity between the control and experimental subjects.

2. sample bias: If one wishes to make generalized statements about populations (and that, after all, is the purpose of experimentation) it is essential that the sample truly represents the population in all its vagaries. This means controlling for factors such as geographical location and 'culture', and using subjects whose performance in the target area of study covers a wide range.

The researcher's task

Using a Posttest-Only Control Group Design, the researcher was to assess the effects of Bidirectional Translation. Half of the subjects would constitute the experimental group, the remainder the control group. The experimental group was to experience

Bidirectional Translation, which would involve the researcher in some induction of teachers in the protocol of the methodology. The control group was to be taught addition and subtraction by whatever means the teachers in that group typically used. After the teaching methodologies had been effected, the researcher was to test a cross section of all the children who had been involved.

Negotiating access to the subjects

Taking full account of the provisos outlined above, the researcher planned and executed her access to a sample of subjects as follows:

1. The researcher sought permission of the Director of Education to pursue her research interests in a number of schools, number as yet unspecified.

2. Negotiations took place between the Glasgow Division Education Officer, the Primary Adviser with responsibility for Mathematics and the researcher; which resulted in agreement that the researcher could approach a manageable number (for the researcher) of schools to solicit their participation.

3. The researcher decided that, given her teaching commitments which, incidentally, radically switched in February 1988 from teaching primary one children to teaching student teachers at the local college of education (to where the researcher had been seconded to lecture in psychology) the total number of

schools which she could comfortably 'manage' would be about twelve.

4. An initial list of about twenty schools was drawn up on the basis of intelligence made available by the local education authority. Such intelligence consisted of the sizes of school rolls; the number of primary one classes in any given school; whether or not the school was in an area of priority treatment; the social and cultural composition of schools; the geographical location of schools and any features which were 'peculiar' to a school. One 'peculiar' feature is, for example, that a small number of schools have a nursery class which is a part of the school whilst, for the most part, nursery education is provided in establishments which are distinct and autonomous from mainstream primary schools. The criteria for the selection of the provisional list of schools from the hundreds which were available to the researcher will be discussed later.

5. Letters were sent to the head teachers of twelve schools. The letters merely indicated the researcher's area of interest and asked permission to visit each school with a view to talks between herself, the head teacher and the primary one teacher. Follow-up telephone calls to secure an invitation to visit, allowed head teachers to accept, or decline the offer of participation in the research without feeling pressurized or 'losing face'. Three head teachers declined on the grounds that they had undertaken other piloting or inservice work and were thus fully committed. For these refusals another three schools

were selected by the researcher. The replacement schools agreed to participate.

6. Initial visits to experimental and control schools were carried out in February 1988. In the experimental schools, the researcher briefly outlined the rationale for Bidirectional Translation, talked the teacher through the guidelines for implementing the methodology, provided the teacher with the necessary materials and arranged to meet again with the teacher to collect feedback and have a chat with the children. In the control schools, the researcher merely indicated that she was collecting data on teachers' views of the teaching of addition and subtraction, had the teacher complete a questionnaire and arranged with the teacher to return to the school towards the end of the school term to have a chat with the children when they would have completed the teacher's usual programme for the teaching of addition and subtraction.

The guidelines and questionnaire mentioned in (6) are appended at the end of this chapter.

7. In May/June the researcher, by arrangement, visited all twelve schools having obtained permission to test 15 children in each school over two successive days for each school.

Factors which were considered

The experimentation which took place in this research endeavoured to secure the best possible balance between internal and external validity. Much of what will be stated in this section has already been implied, but in the interests of precision, the factors constituting internal and external validity will be examined in terms of this piece of research.

to attain internal validity

1. The limitations of this study meant that the variable of history could not be controlled for beyond the most rudimentary levels: to have thoroughly investigated historical events in the subjects' environments would have meant research on a scale which would be impracticable for someone working single-handed. However, a few simple precautions were taken to attenuate the worst excesses of historical contamination. No schools were chosen which had been involved in closures or mergers as a result of local government rationalization in education. No schools were chosen which had experienced a particularly traumatic or dazzling event, such as a fire or a prestigious award/visit. These precautions were thought to eliminate variables such as stress or over-excitement which may have been implicated in the performance measures. A further precaution taken was that only schools which had one primary one class could be included, lest there be any notions of

streaming, however tacit, in operation. If there was only one primary one class in the school, all the primary one children had to be there!

2. The variable of maturation was controlled for by using only primary one children both in experimental and control schools and conducting the testing over a time scale of weeks rather than months.

3. The possibly adverse effects of pre-testing were avoided by not conducting any pre-tests. A two-group design was used instead.

4. The variable of instrumentation was controlled for by the researcher herself doing all the testing and using a predetermined format for this.

5. The variable of subject selection was complex to handle. Constraints of practicality and ethics meant that it was necessary, though not perhaps ideal, to consider **schools** as experimental or control. But since schools vary on a variety of dimensions such as the cultural composition of its population, the ratio of pupils to teachers and the diversity of socio-economic status, it would be very easy to have an experimental group comprising children from advantageous circumstances and a control group comprising children from disadvantageous circumstances, with the resulting findings being heavily contaminated. Given that it was desirable to have a sample which was representative of a population of primary one children, and given that the experimental and control schools

were to be as equitable as possible, the following compromise in the selection of subjects was reached. Factors such as the geographical location of schools (whether inner city or in a peripheral housing estate); the cultural composition of individual school populations (mainly monocultural or mainly multicultural) and the incidence of compensatory provision in schools for 'disadvantaged' pupils (whether or not the school was designated as being in an area of priority treatment) were recognized as being powerful influences on the child's performance in school. These factors were therefore the initial criteria by which schools were selected. These factors do not, however, manifest themselves in isolation. Differing permutations result in schools which can be identified as:

multicultural/A.P.T./inner city schools,
multicultural/non A.P.T./inner city schools,
monocultural/A.P.T./inner city schools,
monocultural/non A.P.T./inner city schools,
monocultural/A.P.T./housing estate schools,
monocultural/non A.P.T./housing estate schools.

The identities of schools having been determined, it was then a matter of selecting two schools of each 'type', one of which was arbitrarily designated as experimental and the other as control.

A further compromise in the selection of subjects was that the subjects tested by the researcher were selected by their class teachers. The researcher requested 15 subjects in each school who:

a) had completed the programme of addition and subtraction;

and

b) were in the teacher's professional opinion 'good' (5 subjects), 'average' (5 subjects) and 'poor' (5 subjects).

This is a somewhat irregular method for selecting subjects for testing but the researcher was at pains not to give participating teachers the impression that it was **they** who were being assessed, and so she considered it prudent, in the circumstances, to have the teachers select the subjects. However, the method comes close to random selection and probably did not damage the experiment.

6. The variable of stability was controlled for by subjecting the test performance data to statistical analyses.

<u>to attain external validity</u>

1. The variable of the Hawthorne Effect was controlled for by having the teachers in the control schools complete a questionnaire on the teaching of addition and subtraction at the outset and securing their agreement for a return visit by the researcher. Neither the experimental nor the control group was made aware of the other's participation in the research, and since each group received the same number of visits from the researcher, each received (at face value) the same amount of researcher intervention.

2. The variable of sample bias has largely been dealt with in what was said about sample selection. Perhaps it should be added, however, that the schools were drawn from north, south, east and west of the city.

The Scope and Aims of the experimentation

The aim was to investigate the effect of Bidirectional Translation on the conceptual understanding of addition and subtraction. Specifically, the experimental hypothesis was that children exposed to the methodology of Bidirectional Translation will perform significantly better on addition and subtraction test items than children who are exposed to traditional methods which place a heavy reliance on computation alone.

The scope of the experiment was restricted to exploring the extent of conceptualization of addition and subtraction. Specifically, the extent to which young children could evoke and utilize addition and subtraction operations was at issue.

Procedure

A list of 20 test items, 10 for evocability and 10 for utilizability, was used as it had been in the Main Study. The number domain used was within 10 and number triples were randomly assigned to test texts, which, similarly, were presented to subjects in random order. The texts of the test items are included at the end of this chapter.

The researcher visited each school on two successive days, on one day to test for evocability and on the other to test for the utilizability of addition and subtraction. Presentation of evocability and utilizability test items were counterbalanced. The researcher worked in a corner of the classroom withdrawing subjects one at a time from the main body of the class. The researcher spent a morning with the class on each visit. Interview time with each subject was about 10 minutes. The instructions to each subject were:

> for evocability
> I'm going to read you a silly story. Listen carefully and tell me if you should add or subtract to find the answer . . . Well done, now let's try some more.
>
> for utilizability
> I'm going to read you a silly story. You see if you can find out the answer and tell me what it is . . . Well done, now let's try some more.

After the subjects made a response, a few seconds were allowed to elapse before another verbal context was presented. The purpose of this was to enable the subject to reflect upon, and possibly change his/her decision. If the subject did make a change in his/her decision, he/she was asked by the researcher to clearly indicate which response the subject believed to be correct.

Each verbal context was read and reread to the subject. Any subsequent lapse of memory on the part of the subject was remedied by the researcher's reading the verbal context yet again.

Successful/unsuccessful performance on each test item was noted.

To conclude the researcher-visits to the schools, the researcher debriefed the participating teachers as follows.

For the experimental teachers the researcher acknowledged their participation and explained that the children seemed to have a sound grasp of what they were doing when they added or subtracted. She further indicated that if the same results were found in other school participating in the research she would have grounds for recommending that the approach of Bidirectional Translation be adopted by other teachers. She also invited teachers' comments and collected what written feedback they had available.

For the control teachers the researcher expressed her thanks for allowing her time to chat with the children and explained that she now hoped to try to develop some method of linking the children's achievements in computation with contexts for addition and subtraction such that children could solve arithmetical problems more easily. She also invited teachers' comments on her 'idea'.

In summary:

(i) 180 subjects, 90 experimental and 90 control, from twelve Glasgow primary schools participated in assessment of their grasp of understanding of addition and subtraction;

(ii) this grasp of understanding was measured in terms of evocability and utilizability;

(iii) the experimental subjects were, before testing, taught to add and subtract using the methodology of Bidirectional Translation;

(iv) the control subjects, before testing, had been taught to add and subtract by whatever was the teacher's usual methodology;

(v) claims are made for the experimental design being both simple and efficient, a Posttest-Only Control Group Design.

Teaching Primary One Children to Add and Subtract with greater understanding: a research investigation.

Guidelines for teachers participating in the investigation.

Effie Maclellan, Division of Education and Psychology, Jordanhill College.

Preface

The research evidence from researchers in Britain and elsewhere indicates that children can learn to add, subtract, multiply and divide when such operations are in the form of 'sums' such as 9+6, 4x2, 6-3. However, many children are unable to

a) translate such an operation into a hypothetical, real world context: in other words they cannot envisage a situation in which such an operation would be required

and conversely

b) extract the appropriate numerical operation from a hypothetical real world context: in other words they have extreme difficulty in solving the word problems as they typically appear in mathematical textbooks.

I believe that unless children can at least to some extent be proficient in (a) and (b) above (or to put it more theoretically, bidirectionally translate between the numerical representation and a verbal context) the concept of addition or subtraction or multiplication or division has not established itself in the child's mind. It is with the aim of introducing the concepts of addition and subtraction in a fuller and more meaningful way that the following methodology has been developed.

At several point in the Series of Steps for teaching Addition and Subtraction, there will be instructions to take out a specific number of cubes. This number is not sacred. Teachers should feel free to choose their own examples using small numbers.

Throughout, frequent reference is made to the terms 'number story' and 'silly story'. This is not meant to be patronizing towards teachers. It is simply an easily understood distinction between formal, notational representation and verbal contextualization of a numerical operation; and in using the terms, 'number story' and 'silly story' the children understand the distinction.

In all of this booklet I have tried to avoid jargon. However there is one exception, found only in step 6 of Subtraction, where reference is made to the terms, minuend and subtrahend - purely because this is the most economical way of explaining what is subtracted from what:

thus 9 minuend
-6 subtrahend
—

The materials needed in using the following methodology are few and simple. The Unifix Cubes typically found in infant classrooms, or their equivalents are perfectly suitable as 'concrete apparatus'. Additionally, the teacher needs some magnetic numerals and operator signs and a magnetic surface on which to use them.

I'd like you to try out the following steps with your different groups of children.

Series of steps for teaching addition

Step 1 Setting the scene.

The group of children is seated round the table, each child having a stack of 10 Unifix Cubes. The children are asked to take two cubes from their stacks. A magnetic numeral '2' is displayed on the magnet board.

2

MAGNET BOARD

Attention is drawn to the two cubes in front of each child and to the numeral '2' on the board. The children are told that they are to take more cubes from their stacks and that to show on the board that they are taking more cubes, a sign is used. The children are told that the sign says 'plus' or 'add on'. The magnetic '+' is affixed to the board.

2 +

MAGNET BOARD

The children are now asked to take a further three cubes from their stacks and to sit them beside the two cubes:

```
  []
[][]
[][]
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A magnetic '3' is displayed on the board.

2 + 3

MAGNET BOARD

Attention is drawn to the cubes in front of the children and to the display on the board. The children are asked to find out how many cubes they took from the stack altogether. After the total has been ascertained, the teacher explains that another sign is needed to show that something has been found out about 2+3. The children are told that the sign says 'equals' or 'makes' or 'is the same as'. The magnetic '=' is affixed to the board as is the magnetic '5'.

2 + 3 = 5

MAGNET BOARD

Attention is drawn to the cubes in front of the children and to the 'number story' on the board (two plus three equals five). The children are invited to 'read' the number story aloud. This entire procedure is repeated many more times over successive days using different number combinations within 10. Zero is introduced by instructing the children to take out 4 cubes and then take out no cubes.

Step 2 Let's Pretend.

The children are introduced to the notion that cubes can be used to represent just about anything in the real world. The teacher says to the children, "Let's pretend the cubes are bananas" or "cars" or whatever. The children are instructed to take out three bananas and then another three bananas and to find out how many bananas they have in front of them. As above, considerable practice is given, and every addition activity is accompanied by its representation in magnetic numeral form.

Step 3 Silly Stories.

The children are told to listen to a 'silly story'. While they are listening they have to take from their stack of cubes the numbers mentioned in the 'silly story'

"Mummy gave me three lollipops and four sweets".

The children are asked to show their three lollipops (whereupon each child holds up the three Unifix Cubes) and their four sweets (whereupon each child holds up the four Unifix Cubes). The teacher asks, "How many things did Mummy give me altogether?" When the total has been identified the teacher asks, "How did you find out the answer?"

Step 4 Silly Stories and Number Stories.

The teacher provides a complete 'silly story':

"There are four blue sweets and two red sweets in the bag so that makes six sweets altogether".

The children are invited to use the magnetic numerals and signs to represent the 'silly story' as a 'number story' (4+2=6). The children 'read' the 'number story' (four plus two equals six) and are required to indicate which number represents the blue sweets, which number represents the red sweets, which sign represents the operation of addition and which sign represents the outcome of the operation. Again, much practice is given in this activity.

Step 5 Number Stories and Silly Stories.

The teacher provides a complete 'number story' on the magnet board (for example 1+3=4) and the children are invited to provide a corresponding 'silly story'. Allow as many children as time allows, to provide verbal contexts for any given numerical representation.

Step 6 Drawing a story - first version.

The children are instructed that instead of telling a 'silly story' they have to draw a 'silly story' for a bit of a number story which will be provided. The instruction 'draw 2+3' is given orally and is also put on the magnet board for the children to see. Provide paper and pencils/pens/ crayons and observe what happens. When each child has pictorially represented his/her 'silly story' ask the child to explain his/her story and scribe the story in front of the child. This procedure of drawing a 'number story' is repeated regularly over successive teaching sessions.

Step 7 Drawing a story - second version.

The children are invited to draw their own 'silly story' with no numerical stimulus being provided. In other words the children are not told of how many of each sub set to draw. There is now greater need than before for the children to describe/ explain their stories to the teacher (since both the numerical components and the verbal contexts are the children's own with no constraints imposed by the teacher) who again scribes at the child's dictation.

Step 8 Strategies for finding the answer.

The children are told that they will be given a bit of a 'number story' (for example 3+4=) and that they will have to find the answer. The teacher asks the children how they will find out the answer if they do not already know. The children make various suggestions:

a) count on their fingers
b) count with cubes
c) draw pictures

each of which is positively received by the teacher who then points out that:

a) sometimes we might not have enough fingers (as when summing any numbers the total of which is greater than 10)
b) cubes are not always available
c) pictures can take a long time to draw.

The teacher the demonstrates a 'method which she sometimes uses'. Whereupon she writes 3+4= on the blackboard and sets out the appropriate number of tally marks:

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 3 + 4 =
||| ||||
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MAGNET BOARD

The results are compared using each of the three methods - fingers, cubes and tally marks. The children are given practice in setting down 'number stories' and in using tally marks (referred to as strokes) but are reassured that each of the methods is valid and that the final choice (of which method to use) is to be theirs.

Step 9 Does it work?

Only now are the children considered ready to undertake the conventional addition exercises of adding two numbers the total of which is within 10. The children undertake this activity outwith the direct supervision of the teacher - that is when she is working with other groups of children in the class. However, to check that the earlier steps in the series have been of use to the children, random, one-to-one interviews are held between the teacher and the child when the teacher in the light of a completed exercise:

a) asks the child how he/she found the answer to a particular operation, say 6+2;

b) invites the children to provide a 'silly story' for a particular addition operation, say 5+0;

c) requests the child to peruse all the examples in the exercise and identify which 'number story' is being referred to when the teacher provides a 'silly story'.

Questions to guide the teacher's observations of the children whilst teaching addition.

Step 1 How do the children find the 'altogether' number. Do they

a) use cubes,if so how many?
b) count on their fingers, if so how many?
c) make use of number facts, if so how many?
d) other?

Step 2 What comments, if any, do the children make when they are pretending that the Unifix Cubes are objects in the real world?

Step 3 Can the children report how they performed the addition operation? Do they make reference to strategies such as those outlined in Step 1?

Step 4 What hesitancies/difficulties, if any, do the children have when using the magnetic numerals/signs to record a 'story'. For example, do the children use the operator signs effectively?

Step 5 Please include, about 6, transcriptions of children's 'silly stories'.

Step 6 a) Do the children insert the operator signs appropriately?
b) Do the children complete the addition operation of their own volition?

Step 7 as for Step 6

Step 8 a) What suggestions do the children make as to how the answer can be obtained?
b) How do the children respond to the tally marks?

Step 9 Please comment on the children's performance in the one-to-one interviews for 9 (a), (b) and (c).

Series of steps for teaching subtraction

Step 1 Setting the scene.

The group of children is seated round the table, each child having a stack of 10 Unifix Cubes. The children are asked to take 6 cubes from their stacks. A magnetic numeral '6' is displayed on the magnet board.

6

MAGNET BOARD

Attention is drawn to the six cubes in front of each child and to the numeral '6' on the board. The children are told that they are to take some cubes away from their set of six and that to show on the board that they are taking cubes away a sign is used. The children are told that the sign says 'minus' or 'subtract' or 'take away'. The magnetic '-' is affixed to the board.

6 -

MAGNET BOARD

The children are now asked to take two cubes away from their stack of six and to return them to the 'bank'. A magnetic '2' is placed on the board.

6 - 2

MAGNET BOARD

Attention is again drawn to the board which now displays 6-2. The children are asked to find out how many cubes they have left. After the answer has been ascertained the teacher reminds the children that a sign is needed to show that something has been found out about 6-2. The magnetic '=' is affixed to the board as is a magnetic '4'.

6 - 2 = 4

MAGNET BOARD

The children are reminded that they started off with six cubes and that they took away two of them. They are now left with four cubes in front of them. Attention is drawn to the 'number story' on the board (6-2=4) and the children are invited to 'read' the 'number story' (six minus two equals four) aloud.

The entire procedure is repeated many more times over successive days using different number combinations within ten. Zero is used by instructing the children to take out eight cubes and then take away no cubes.

Step 2 Let's Pretend.

The children are reminded that cubes can be used to represent anything in the real world. The teacher tells the children, "let's pretend the cubes are dogs" or "houses" or whatever. The children are instructed to take out five dogs and then to take three dogs away and to find out how many dogs are left. As before, considerable practice is given and every subtraction activity is accompanied by its representation in numerical form.

Step 3 Silly Stories.

The children are told to listen to the 'silly story' and to operate with the cubes accordingly:

"Mummy had four apples but she gave me one to eat".

The children are asked to show their four apples (whereupon each child holds up his/her four Unifix Cubes) and to show that one was eaten (whereupon each child demonstrates the subtraction). When the children correctly identify how many apples are left, they are asked by the teacher how they found out the answer.

Step 4 Silly Stories and Number Stories.

The teacher provides a complete 'silly story':

"Three cups were on the shelf. One of them got knocked on to the floor so that left only two cups".

The children are invited to use the magnetic numerals and signs to represent the 'silly story' as a 'number story' (3-1=2). The children 'read' the 'number story' (three minus one equals two) and are required to indicate which number represents the cups at the beginning of the story, which number represents the cup that met with the accident, which number represents the cups at the end of the story, which sign represents the operation of subtraction and which sign represents the outcome of the operation. Many such verbal contexts are provided by the teacher.

Step 5 Number Stories and Silly Stories.

The teacher provides a complete 'number story' on the magnet board (for example 7-4=3), and the children are invited to provide a corresponding 'silly story'. Allow as many children, as time allows, to provide verbal contexts for any given numerical representation.

Step 6 Drawing a story - first version.

The children are instructed that instead of telling a 'silly story' they will have to draw a 'silly story' for a bit of a 'number story' which will be provided. The instruction 'draw 5-3' is given orally and is also put on the magnet board for the children to see. Provide paper and pencils/pens/crayons and observe what happens. Some teacher intervention may be required because the children may pictorially represent the five and also the three and proceed to add rather than subtract. If this happens, still request the child to explain/describe his/her 'silly story'. If required to reflect on their own story the children may not be too happy with a story in which they have five sweets, eat three of them and then be left with eight! Ask the children how they can show on paper the initial quantity, the operation and the result. Some may suggest rubbing out the subtrahend. This is perfectly reasonable but in so doing:
a) nobody will be able to see how many things were present at the beginning of the story;
b) nobody will be able to see how many things were taken away.
By emphasizing that the minuend and the subtrahend both be visually evident, hopefully some child or children will suggest that the minuend is drawn and that the subtrahend is represented by crossing out. Considerable practice is again required.

Step 7 Drawing a story - second version.

The children are invited to draw their own 'silly story' with no numerical stimulus being provided. In other words the children are not told how many things to draw initially or how many to score out. They must, however, verbally report the context in order that it can be written down by the teacher.

Step 8 Strategies for finding the answer.

The children are told that they will be given a bit of a 'number story' (for example 7-2=) and that they will have to find the answer. The teacher asks how they will find out the answer if they do not already know. Hopefully the children will suggest one/all of the following:

a) use their fingers
b) use cubes
c) use strokes

The teacher checks that the children are able to use all of these strategies.

Step 9 Does it work?

Only now are the children considered ready to undertake conventional subtraction exercises, within ten. The children undertake this activity outwith the direct supervision of the teacher, who is meanwhile working with other groups of children. However, as with addition, random, one-to-one interviews are held between the teacher and the individual child to check if the earlier steps in the series have been of use to the child. In each interview, and in the light of a completed subtraction exercise, the teacher:

a) asks the child how he/she found the answer to a particular subtraction operation, say 8-3;

b) invites the child to provide a 'silly story' for a particular subtraction operation say 2-2;

c) requests the child to peruse all the examples in the exercise and identify which 'number story' is being referred to when the teacher provides a 'silly story'.

Questions to guide the teacher's observations of the children whilst teaching subtraction.

Step 1 How do the children find what is 'left' after subtracting. Do they

a) use cubes,if so how many children do?

b) count on their fingers, if so how many?

c) make use of number facts, if so how many?

d) other?

Step 2 What comments, if any, do the children make when they are pretending that the Unifix Cubes are objects in the real world?

Step 3 Can the children report how they performed the subtraction operation? Do they make reference to strategies such as those outlined in Step 1?

Step 4 What hesitancies/difficulties, if any, do the children have when using the magnetic numerals/signs to record a 'story'. For example, do the children use the operator signs effectively; do they order the numbers in the same sequence as in the 'silly story'?

Step 5 Please include, about 6, transcriptions of children's 'silly stories'.

Step 6 a) Do the children complete the subtraction operation of their own volition?

b) If the children depict the minuend and the subtrahend as two distinct entities do they insert the operator signs appropriately, and, if so do they

c) complete the subtraction operation appropriately?

Step 7 as for Step 6

Step 8 What suggestions do the children make as to how the answer can be obtained?

Step 9 Please comment on the children's performance in the one-to-one interviews for 9 (a), (b) and (c).

Questionnaire on the teaching of Addition and Subtraction Operations to Primary One Children

1. Are you using a commercially produced Maths Programme or Scheme? YES NO
If yes, which one?

2. Do you have children in your class who came to school already able to add? YES NO

3. Do you have children in your class who came to school already able to subtract? YES NO

4. Do you aim to teach automatic recall of addition number bonds? YES NO

5. Do you aim to teach automatic recall of subtraction number bonds? YES NO

6. Do you prefer to teach addition first and then subtraction? YES NO

7. Do you prefer to teach addition and subtraction as converse operations? YES NO

8. Do you explain the '+' sign as 'plus'? YES NO

9. Do you explain the '+' sign as 'and'? YES NO

10. Do you explain the '+' sign as 'add on'? YES NO

11. Do you explain the '-' sign as 'minus'? YES NO

12. Do you explain the '-' sign as 'subtract'? YES NO

13. Do you explain the '-' sign as 'take away'? YES NO

14. Do you explain the '=' sign as 'equals'? YES NO

15. Do you explain the '=' sign as 'the same as'? YES NO

16. Do you allow children to use their fingers to find the answer? YES NO
Please give a reason for your point of view.

17. Do you approve of children using their fingers to find the answer? YES NO
Please give a reason for your point of view.

18.Do you 'embed' the addition operation in a verbal context - i.e. provide a word problem? YES NO
Please give a reason for your answer.

19. Do you 'embed' the subtraction operation in a verbal context? YES NO
Please give a reason for your answer.

20. In your professional opinion is Maths teaching best kept as a distinct curricular area?
YES NO
Please give a reason for your answer.

Items to test the conceptualization of Addition and Subtraction

The Utilizability of Addition

Joining: Betty has (x) dollies. Granny gave her (y) more dollies. How many dollies has Betty got now?
Separating: Fiona had some carrots. She gave (x) carrots to the rabbit and now she has (y) carrots left. How many did she have to start with?
Part-Part-Whole: (x) girls and (y) boys went out to play in the playground. How many children went out to play?
Comparison: Susan has (x) hats. Mummy has (y) more hats than Susan. How many hats has Mummy got?
Equalizing: There were (x) red cars in the car park. (y) more red cars came in. Now there are the same number of red and blue cars in the car park. How many blue cars are there?

The Utilizability of Subtraction

Joining: Mr Brown has (x) shirts. How many more shirts does he need to have (z) shirts altogether?
Separating: John had (z) pencils. He gave(x) to his big brother. How many pencils does John have now?
Part-Part-Whole: Daddy has (z) saws. (x) of them are very blunt and the rest are very sharp. How many very sharp saws does he have?
Comparison: There are (z) men and (x) women standing at the bus stop. How many more men are at the bus stop?
Equalizing: There are (z) forks and (x) knives in the drawer. How many forks should I take out so that there are the same number of forks and knives in the drawer?

The Evocability of Addition

Joining: Tom has (x) bananas. Susan gave him (y) more bananas. How many bananas has Tom got now?
Separating: Neil had some toys. He gave (x) toys to his little brother and now he has (y) toys left. How many did he have to start with?
Part-Part-Whole: There are (x) girls and (y) boys in the room. How many children are in the room altogether?
Comparison: Mary has (x) cats. Christine has (y) more cats than Mary. How many cats does Christine have?
Equalizing: There were (x) boys in the playground. (y) more went out. Now there are the same number of boys and girls in the playground. How many girls are in the playground?

The Evocability of Subtraction

Joining: Karen has (x) lollipops. How many more lollipops does she need to have (z) lollipops altogether?
Separating: Imran had (z) sweets. He gave (x) to Linda. How many sweets does Imran have now?

Part-Part-Whole: Eva has (z) pens for colouring with. (x) of them are blue and the rest are red. How many red pens has Eva got?
Comparison: There are (z) girls and (x) boys in the cloakroom. How many more girls are in the cloakroom?
Equalizing: There are (z) cups and (x) saucers on the table. How many saucers should I take away so that there are the same number of cups and saucers on the table?

RESULTS

The purpose of a chapter such as this is twofold. First, it presents the results of the study in a summary form and then it points out some patterns observed within the data. In both of these processes interpretation is taking place. The summary involves data reduction and it is necessary to show that the categories or dimensions implied in this process are reasonable ones. Here descriptive statistics can provide rigour. The process of extracting patterns involves making distinctions and comparisons and again statistical tests can ensure that the author's interpretation is true to the data.

The remainder of this chapter will thus attempt to describe, and generalize from:

(i) how the subjects performed on the different tasks;

(ii) variations and associations within the data;

(iii) differences between the experimental and control subjects.

The data being dealt with in this chapter will be considered to be on an interval scale. However, it is acknowledged that this involves an assumption common in dealing with test results in education and psychology; that the correct answer to any one item is exactly equivalent (in terms of the 'amount' of the underlying competence expressed) to the correct answer for any other item.

The dubiety expressed about the scale of measurement is not merely cautionary, but has implications for the type of statistical tests used on the data. Statisticians of the 'purist' school would claim that at least an interval scale is a necessary condition for the use of parametric tests. Statisticians of the more 'liberal ' school would, on the other hand, claim that parametric tests are so robust that they can still be applied even if the equal units in the interval scale are more quasi than real. For the researcher who is naive in statistical theory, it is not clear which type of advice one should follow. As a result, the position being adopted here is as follows.

The data are assumed to be on an interval scale. Having said that however, it does not necessarily mean that the tests used will be exclusively parametric, since another necessary condition for the use of parametric tests is that the results be normally distributed. As will be seen later this assumption is not met in the data in this study. But here again, there is some disagreement amongst the experts. Some would claim that parametric tests should not be used if the data are skewed, others would claim that there can be considerable deviation from normality without the result of the parametric test being affected.

While the non parametric tests are not so powerful as their parametric counterparts (that is, they can 'lose' valuable information in the data), they nevertheless have greater generality. To use a parametric test inappropriately can 'add' information which is not justified. Although the parametric test can detect significance where a non parametric test might not do so, the corollary to that is that if a non parametric test does find significance so must its parametric counterpart.

Accordingly then, non parametric tests will be used principally, and only in instances where there seems further cause to tease out the data, will parametric tests be used.

SUMMARY OF OBSERVATIONS

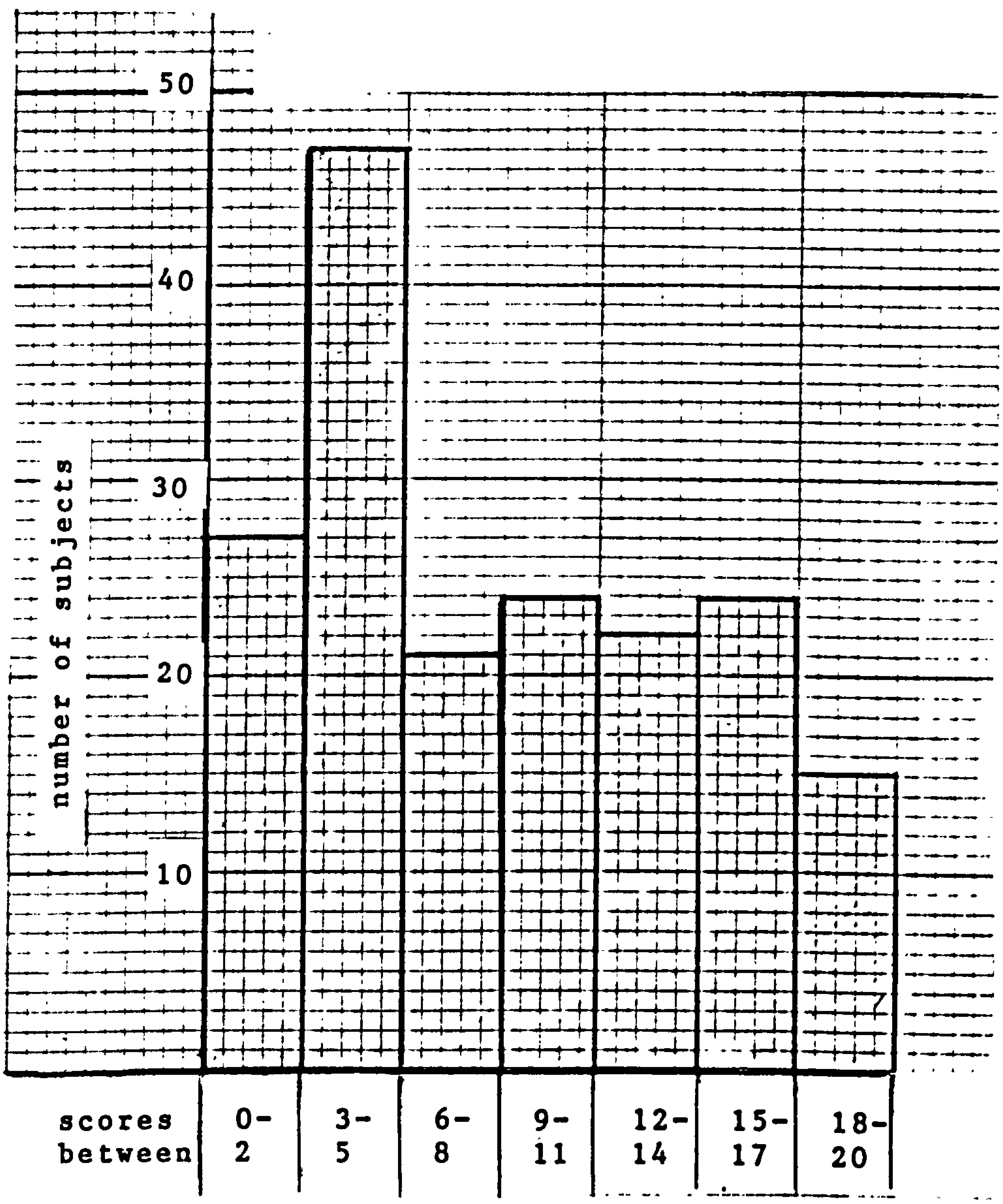
There were 90 experimental subjects and 90 control subjects. Overall, 20 responses for each subject were scored.

Table 9.1 Frequency Distribution of Class Intervals of Correct Scores for 180 Subjects

scores between	experimental frequencies
18 - 20	15
15 - 17	24
12 - 14	22
9 - 11	24
6 - 8	21
3 - 5	47
0 - 2	27
Number	180
Mode	4
Median	8
Mean	8.75
S.D.	5.89
Sk.	0.38

The frequency distribution can also be depicted graphically as the following bar graph shows.

Figure 9.1 Bar Graph to show Frequency of Scores



Overall frequency distributions are of limited interest given that the test items were not all of one type. The following table shows the number of correct responses (in percentage terms) to each type of item within each class of verbal context.

Table 9.2 Percentage Matrix of Correct Responses to the different items for 180 Subjects

Type of concept.	Class of Verbal Context					
	Join	Sepa	Part	Comp	Equa	MEANS
Util +	87.8	64.4	94.4	40.6	36.1	64.7
Util -	42.2	84.4	67.8	21.7	39.4	51.1
Evoc +	48.9	12.8	56.7	28.3	27.2	34.8
Evoc -	7.2	47.8	37.8	5.6	23.3	24.3
MEANS	46.5	52.4	64.2	24.1	31.5	43.7

Visual inspection of Table 9.2 shows the numbers of correct responses for <u>all</u> of the subjects and conceals the difference in performance between the experimental and control subjects. So what can be gleaned from Table 9.2 is information regarding overall mean performance, and as such is not of great interest. On average more subjects responded correctly when they were required to utilize concepts than when they were required to evoke concepts. More subjects responded correctly when addition was at issue than when subtraction was at issue. And finally, amongst the different classes of verbal context, more subjects responded correctly to the Part-Part Whole class of verbal context than to any other.

This variability in correct response can be more specifically described by tabulating the percentage of correct respondents for each of the twenty items. Again, however, these figures do not distinguish between experimental and control subjects.

Table 9.3 Percentage of Correct Respondents for each item (for 180 subjects in rank order)

Item	% Correct
(P)UTIL+	94.4
(J)UTIL+	87.8
(S)UTIL-	84.4
(P)UTIL-	67.8
(S)UTIL+	64.4
(P)EVOC+	56.7
(J)EVOC+	48.9
(S)EVOC-	47.8
(J)UTIL-	42.2
(C)UTIL+	21.7
(E)UTIL-	36.1
(P)EVOC-	37.8
(E)UTIL+	36.1
(C)EVOC+	28.3
(E)EVOC+	27.2
(E)EVOC-	23.3
(C)UTIL-	21.7
(S)EVOC+	12.8
(J)EVOC-	07.2
(C)EVOC-	05.6

key to reading Table 9.3 The first letter (in brackets) represents the class of verbal context: P = Part-Part Whole, J = Joining, S = Separating, C = Comparison and E = Equalizing. The letters UTIL and EVOC represent the utilizability and evocability of concepts and the + and - signs are for addition and subtraction.

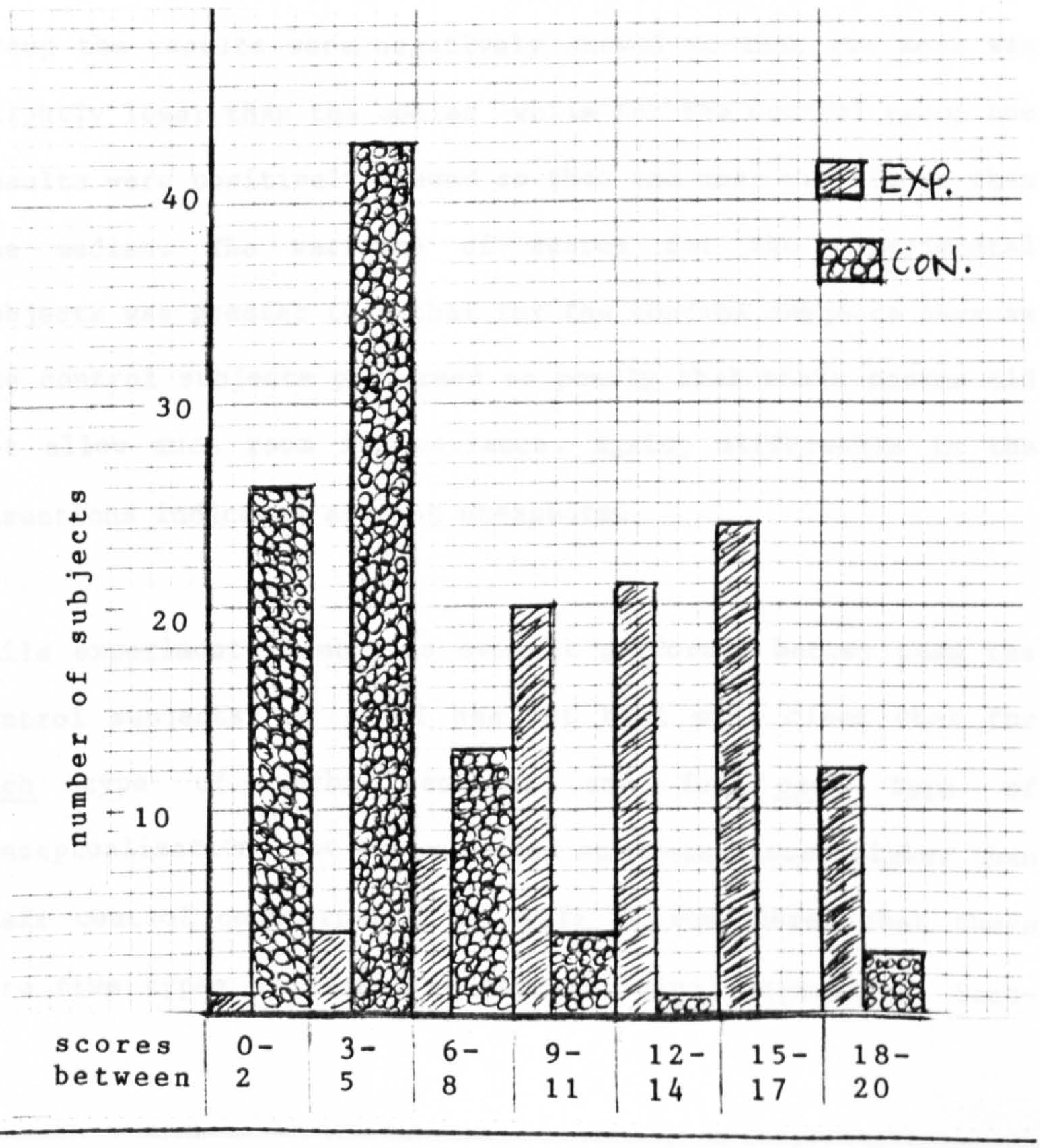
HOW THE SUMMARY DIVIDES FOR EXPERIMENTAL AND CONTROL SUBJECTS

The above figures have given only a global picture of task performance, which is of minor importance in this study. It is more important to see how the experimental and control subjects performed separately.

Table 9.4 Frequency Distribution of Class Intervals of Correct Scores for 180 Subjects

scores between	experimental frequencies	control frequencies
18 - 20	12	3
15 - 17	24	0
12 - 14	21	1
9 - 11	20	4
6 - 8	8	13
3 - 5	4	43
0 - 2	1	26
Number	90	90
Mode	14	4
Median	14	4
Mean	13.122	4.378
S.D.	4.354	3.514
Sk.	-0.61	0.32

Figure 9.2 Bar Graph to show Frequency of Scores for experimental and control subjects



From inspection of the data above, the findings are very much as anticipated, and as we shall see, the patterns within them are consistent. The results indicate that experimental subjects performed better than the control subjects, with the scores of the experimental subjects bunching towards the top end of the scale but the scores of the control subjects bunching towards the bottom end of the scale. This strong difference favours the methodology of Bidirectional Translation.

For each group of cases the modal and median values were the same but the mean value was different. For the experimental group the results were negatively skewed so that the mean was slightly lower than the median while for the control group the results were positively skewed so that the mean was higher than the median. The variance of scores for the experimental subjects was greater than that for the control subjects because the control subjects performed so poorly that their scores did not allow much room for variance. Again, differences in the directions indicated are not unexpected.

While experimental subjects overall performed better than the control subjects, it still has not been made clear that for <u>each</u> type of verbal context and for <u>each</u> type of conceptualization, the experimental subjects scored higher than their control counterparts. It will be remembered that there were five types of verbal context (Joining, Separating, Part-

Part Whole, Comparison and Equalizing) and four types of conceptualization (utilizability of addition, utilizability of subtraction, evocability of addition and evocability of subtraction). The frequencies of occurrence for each is tabulated below.

Table 9.5 Matrix of Correct Responses for 90 Experimental (Exp) and 90 Control (Con) Subjects

Type of concept.	Join		Sepa		Part		Comp		Equa	
	Exp	Con	Exp	Con	Exp	Con	Exp	Con	Exp	Con
Util +	89	69	80	36	89	81	62	11	48	17
Util -	60	16	87	65	85	37	34	5	60	11
Evoc +	82	6	20	3	87	15	48	3	45	4
Evoc -	12	1	80	6	64	4	10	0	39	3

Class of Verbal Context

Table 9.6 Summarizing Totals from Table 9.5

Column Totals (Class of Context)	Exp	Con	Row Totals (Type of Context)	Exp	Con
Join	243	92	Util +	368	214
Sepa	267	110	Util -	326	134
Part	325	137	Evoc +	282	31
Comp	154	19	Evoc -	205	14
Equa	192	35			
Totals	1181	393		1181	393

Inspection of Tables 9.5 and 9.6 allows the following observation:

For the five classes of verbal context, the order of performance was Part-Part-Whole > Separating > Joining > Equalizing > Comparison. This trend was the same for both experimental and control subjects. The same similarity was found for the types of conceptualization where the order of performance was the utilizability of addition > the utilizability of subtraction > the evocability of addition > the evocability of subtraction.

As might be expected, the correct response to the different test items was variable. Below follows a listing of item means.

Table 9.7 Item Means for experimental and control subjects in rank order

Exp. Subjs. Item	Exp. Subjs. Mean	Con. Subjs. Item	Con. Subjs. Mean
(P)UTIL+	0.989	(P)UTIL+	0.900
(J)UTIL+	0.989	(J)UTIL+	0.767
(S)UTIL-	0.967	(S)UTIL-	0.722
(P)EVOC+	0.967	(P)UTIL-	0.411
(P)UTIL-	0.944	(S)UTIL+	0.400
(J)EVOC+	0.911	(E)UTIL+	0.189
(S)UTIL+	0.889	(J)UTIL-	0.178
(S)EVOC-	0.889	(P)EVOC+	0.167
(P)EVOC-	0.711	(C)UTIL+	0.122
(C)UTIL+	0.689	(E)UTIL-	0.122
(J)UTIL-	0.667	(J)EVOC+	0.067
(E)UTIL-	0.667	(S)EVOC-	0.067
(E)UTIL+	0.533	(C)UTIL-	0.056
(C)EVOC+	0.533	(P)EVOC-	0.044
(E)EVOC+	0.500	(E)EVOC+	0.044
(E)EVOC-	0.433	(E)EVOC-	0.033
(C)UTIL-	0.378	(C)EVOC+	0.033
(S)EVOC+	0.222	(S)EVOC+	0.033
(J)EVOC-	0.133	(J)EVOC-	0.011
(C)EVOC-	0.111	(C)EVOC-	0.000

key to reading Table 9.7 The first letter (in brackets) represents the class of verbal context: P = Part-Part Whole, J = Joining, S = Separating, C = Comparison and E = Equalizing. The letters UTIL and EVOC represent the utilizability and evocability of concepts and the + and - signs are for addition and subtraction.

Perusal of Table 9.7 shows that at the very top and the very bottom of the listings, the rank orderings are the same. In total 8 of the items occupy the same ranks for each group of subjects but the remaining 12 are slightly different.

VARIATIONS WITHIN THE DATA

The rank ordering of item means (see Table 9.7) suggested a similarity of pattern for both the experimental and control subjects. In order to test this null hypothesis that the relative percentages of correct response to each item was a function of chance and nothing whatsoever to do with the nature of the items, the data in Table 9.7 were subjected to the Spearman Rank-Difference Test. The rho coefficient was 0.955, significant at the 1% level (two tailed). This high positive correlation between the ranks is not surprising given the 'parallel' performance between experimental and control subjects observed in Tables 9.5 and 9.6.

It did seem to be emerging from the data that conceptualization could be inhibited or facilitated by the type of verbal context which the subjects were working within at any one time. The matrix of correct responses (see Tables 9.5) suggested that the different classes of verbal context could be ordered in terms of difficulty. To test the null hypothesis that there was no significant difference between the different classes and that differences were a function of chance, the Page's L Trend Test was used. For both the experimental ($L=206.5$, $p<0.01$) and control ($L=211.5$, $p<0.001$) groups the null hypothesis was rejected. Overall, there was a trend showing that subjects performed best on Part-Part-Whole items, followed by

Separating, Joining and Equalizing items. The subjects performed least well on Comparison items.

While the Page's L Trend Test demonstrated a global trend, it was not clear if this trend held for each type of conceptualization. In other words, did the trend hold for the utilizability of addition, the utilizability of subtraction, the evocability of addition and the evocability of subtraction? So to test the null hypothesis that for each type of conceptualization there was no significant difference between the different classes of context and that differences were a function of chance, Chi-Square Tests were run on the data from each type of conceptualization

In the experimental group, for each type of conceptualization, the same trend was found (significant at the 1% level), that the Part-Part-Whole Class was easiest followed by the Separating Class, the Joining Class, the Equalizing Class and finally the Comparison Class. In the control group the trend was repeated in the utilizability of concepts, and in the evocability of addition. It was not possible to run the Chi-Square Test for the evocability of subtraction since the expected frequencies in this instance were less than five, at which point the computed value for χ^2 would have been unstable.

Table 9.8 Chi-Square values for different types of conceptualization in experimental subjects

Type of concept	χ^2 values	d.f.	significance level
Util +	17.73	4	p <0.01
Util -	29.01	4	p <0.01
Evoc +	55.26	4	p <0.01
Evoc -	93.99	4	p <0.01

Table 9.9 Chi-Square values for different types of conceptualization in control subjects

Type of concept	χ^2 values	d.f.	significance level
Util +	90.39	4	p <0.01
Util -	89.53	4	p <0.01
Evoc +	16.58	4	p <0.01

Clearly then, there <u>is</u> a difference between the different types of verbal context. Intuitively, this seems to make sense. The Equalizing Class had items which, by definition, had to be wordy and thus made perhaps rather heavy demands on the individual's processing capacity. Similarly, the Comparison Class had items which traditionally teachers have known (Floyd et al, 1982), and which research (Donaldson & Balfour, 1968; Donaldson & Wales, 1970) has substantiated, can be problematic for young children.

ASSOCIATIONS WITHIN THE DATA

Given the earlier theorizing in chapter three, on the close relationship between cognition and metacognition and the possibility that utilizability and evocability of concepts could be manifestations of these two strands of conceptualization, it seemed appropriate to explore the relationships between the utilizability and evocability scores. Was there any association between the utilizability of addition and the utilizability of subtraction, between the evocability of addition and the evocability of subtraction, between the utilizability and the evocability of addition, between the utilizability and the evocability of subtraction? To test the null hypothesis that the utilizability and evocability of concepts were essentially unrelated, the Chi-Square Test of Association was used on the data.

Table 9.10 Chi-Square values for different types of conceptualization in experimental subjects

Type of concept	χ^2 values	d.f.	significance level
Util +/-	13.05	4	$p < 0.02$
Evoc +/-	109.78	4	$p < 0.001$
Util/Evoc +	27.41	4	$p < 0.001$
Util/Evoc -	26.48	4	$p < 0.001$

For the control subjects the Chi-Square Test could only be computed for the utilizability of addition and subtraction (since in the other types of conceptualization expected frequencies fell below five), where it was found $\chi^2=45.32$, $p<0.001$.

The above findings are quite interesting. There is significant association between utilizability and evocability of addition and subtraction concepts, this association being particularly noticeable for subjects who have been exposed to a teaching methodology which puts emphasis on talking about what one is doing.

DIFFERENCES BETWEEN THE EXPERIMENTAL AND CONTROL SUBJECTS

This entire research study was mounted in the hope of finding a more effective means, than seems currently available, of teaching children to add and subtract with greater understanding. The acid test would be whether or not children exposed to Bidirectional Translation performed any better than those who had not been. The raw data clearly showed differences between experimental and control groups but were these differences really significant? To test the null hypothesis that differences found between experimental and control subjects were due to chance and nothing whatsoever to do with the independent variable of Bidirectional Translation, the Wilcoxon Test should (according to the introductory rationale) have been run on the data. However, visual inspection of the data showed that within each type of verbal context, and for each type of conceptualization, the experimental subjects almost invariably scored higher than their control counterparts which would cause a preponderance of ranks of one sign, leading to significance. Since the difference was obviously significant, it seemed reasonable to seek the size of the difference. To do this, the correlated t-test was needed. A further argument in defence of the t-test at this point is that in a large-sample case (large meaning more than eight pairs of scores) the violation of the assumption of normality becomes less important since the sampling distribution of the Wilcoxon T statistic itself approaches the normal distribution. Thus t values were computed.

Table 9.11 Differences between experimental and control subjects

class/concept	t value	significance
(J)UTIL+	4.81	0.000
(J)UTIL-	7.60	0.000
(J)EVOC+	21.05	0.000
(J)EVOC-	3.24	0.002
(S)UTIL+	7.92	0.000
(S)UTIL-	4.78	0.000
(S)EVOC+	3.94	0.000
(S)EVOC-	19.33	0.000
(P)UTIL+	2.64	0.010
(P)UTIL-	9.27	0.000
(P)EVOC+	18.25	0.000
(P)EVOC-	12.63	0.000
(C)UTIL+	9.43	0.000
(C)UTIL-	5.67	0.000
(C)EVOC+	8.90	0.000
(C)EVOC-	3.34	0.001
(E)UTIL+	5.12	0.000
(E)UTIL-	8.95	0.000
(E)EVOC+	7.95	0.000
(E)EVOC-	7.16	0.000

key to reading Table 9.11 The first letter (in brackets) represents the class of verbal context: P = Part-Part-Whole, J = Joining, S = Separating, C = Comparison and E = Equalizing. The letters UTIL and EVOC represent the utilizability and evocability of concepts and the + and - signs are for addition and subtraction.

The differences between the experimental and control subjects for every class of verbal context and every type of conceptualization are very clear and strongly support the value of Bidirectional Translation. They are highly significant and, at times, of considerable magnitude. This was particularly noticeable for some instances of evocability of concepts.

While the differences between experimental and control groups seem to give some credence to the hypothesis that Bidirectional Translation does enable children to conceptualize addition and subtraction in a more comprehensive fashion than other methods do (insofar as only 3 items produced scores above 50% amongst control subjects whilst 15 of the 20 items produced scores of at least 50% amongst the experimental subjects, as can be seen in Table 9.7), it cannot be claimed that Bidirectional Translation 'solves' the problem of teaching addition and subtraction. Even <u>with</u> the methodology of Bidirectional Translation, 5 of the items produced scores of less than 50%. This, in turn, raises questions as to how appropriate it is for us as teachers to address ourselves to all the types of verbal context which were used in the study. Given the comments earlier, on the complexity of the Comparison and Equalizing Classes, it may be that such items are inappropriate at such an early stage of schooling. But how are we to make decisions as to what to include and what to exclude?

If we were to adopt as a criterion of mastery, that when 80% of the class is successful in a given content area the concepts and skills involved therein are deemed broadly 'suitable' for the learners in question, then we would not include Comparison and Equalizing Classes, both of which fell below a 70% success rate for the experimental subjects, and below 20% for the control subjects. On an 80% criterion of mastery it can be seen from Table 9.7 that 8 of the items were achieved using Bidirectional Translation whereas only 1 item was achieved using alternative methods. So even if Bidirectional Translation cannot 'solve' the problem of teaching addition and subtraction, it nevertheless seems to have some 'power' in the teaching process.

DIFFERENCES BETWEEN ADDITION AND SUBTRACTION

It was a deliberate strategy in this research to teach addition and subtraction operations separately, for the reason given earlier. Opinion is divided on this. Because the operations are complementary, some teachers would advocate their being taught simultaneously. This view presupposes that because the adult/sophisticated learner understands the operations to be related so will the novice learner. If the novice learner does conceive of the operations as being complementary, then, logically, there should be no difference, at least in terms of evocability (where computation is not required), between performance on addition and performance on subtraction. To test the null hypothesis that differences between the evocability of addition and the evocability of subtraction were random, t-tests were computed on the data.

Table 9.12 Differences between addition and subtraction in terms of evocability (exp. subjs.)

class/concept	t value	2 tail prob.
Join	17.65	0.000
Sepa	-12.18	0.000
Part	5.23	0.000
Comp	7.72	0.000
Equa	1.35	0.181

Table 9.13 Differences between addition and subtraction in terms of evocability (cont. subjs.)

class/concept	t value	2 tail prob.
Join	2.29	0.025
Sepa	-1.35	0.181
Part	3.52	0.001
Comp	1.75	0.083
Equa	1.00	0.320

Both the experimental and control subjects found addition easier than subtraction in all classes of verbal context except the Separating Class. However, the difference was not significant in the Equalizing Class for the experimental subjects and the differences were only significant for the control subjects in the Joining and Part-Part-Whole Classes.

A somewhat similar pattern was found when differences between the utilizability of addition and the utilizability of subtraction were found.

Table 9.14 Differences between addition and subtraction in terms of utilizability (exp. subjs.)

class/concept	t value	2 tail prob.
Join	6.50	0.000
Sepa	-2.39	0.019
Part	2.03	0.045
Comp	6.34	0.000
Equa	-2.64	0.010

Table 9.15 Differences between addition and subtraction in terms of utilizability (cont. subjs.)

class/concept	t value	2 tail prob.
Join	11.29	0.000
Sepa	-5.48	0.000
Part	9.23	0.000
Comp	2.52	0.013
Equa	1.51	0.134

Both experimental and control subjects found subtraction easier than addition in the Separating Class and, additionally, the experimental subjects found subtraction easier than addition in the Equalizing Class. Apart from the Equalizing Class for the control subjects, all other differences were significant.

The consistent finding is that addition is easier than subtraction except in the case of the Separating Class (in all types of conceptualization, for all subjects) and in the case of the Equalizing Class (for experimental subjects in the utilizability condition). There is no clear explanation for this. Perhaps in the case of the Separating Class, the only possible hint lies in what the subjects had said to the researcher in the Main Study (see Chapter 7) when a similar phenomenon had occurred. There, the subjects had performed very badly on Separating (evocability +) items because they had been influenced by the semantics of the verbal context to the detriment of its logical structure. By default, then, they had performed better on subtraction items. But this explanation is only partial insofar as it may account for the evocability condition. Perhaps the only reasonable explanation, in the absence of further information, is that in the few cases where the subjects had found subtraction easier than addition they had somehow given greater prominence to cues such as 'gave' and 'take out' than they had in other instances, and that sometimes this corresponded to the correct answer and sometimes it did not.

It would appear then, that by whatever method young children are taught to add and subtract, they do not experience the complementary operations as being of equal ease/difficulty.

DIFFERENCES WITHIN THE EXPERIMENTAL GROUP AND WITHIN THE CONTROL GROUP

While differences between the experimental group and control group were obvious, what was not so immediately clear was if there was any overall difference **within** the experimental group and, again, **within** the control group. There were 6 experimental schools and 6 control schools. Tabulated below are the total number of correct responses for each school.

Table 9.16 Total Number of Correct Responses for each school out of a possible 300

School	Experimental	Control
A	259 (86.3%)	89 (29.7%)
B	156 (52%)	39 (13%)
C	196 (65.3%)	77 (25.7%)
D	183 (61%)	59 (19.7%)
E	188 (62.7%)	68 (22.7%)
F	200 (66.7%)	61 (20.3%)

Visual inspection of Table 9.16 clearly shows variation in performance both among the experimental and control schools, but again, are these differences significant? To test the null hypothesis that the sample schools within each group were from, respectively, the same populations the Kruskal-Wallis test was applied to the data.

Within the experimental group there was a significant difference ($H = 21.44$, $p<.001$). This difference was an overall difference amongst the samples in the experimental group, and while it does not indicate which pairs of samples were significantly different from each other, the fact that there was an overall difference is justification for further analysis of pairs of samples (Siegel, 1956). The Mann-Whitney test showed School A to be significantly different from each of the others (for example, as compared with School F - which was second in terms of performance - the value of U was 52.5, significant at the probability level of .02, two-tailed). What this difference is caused by is not clear. It could be that School A performed better because the children were 'brighter', or the teacher was more effective, or some combination of both. Nor can it be excluded that some other, indeterminate environmental factors may have been at work. School A in the experimental group was the 'matched pair' of School A in the control group, and inspection of Table 9.16 shows School A in the control group also to demonstrate 'best performance'. Similarly, for both experimental and control groups School B demonstrated 'poorest performance'.

Within the control group the difference in performance amongst the six schools was not significant. Within group variations are, then, clearly less than differences between the experimental and control groups. This would appear to be attributable to Bidirectional Translation.

In Summary

(i) the sample of children exposed to Bidirectional Translation performed significantly better on number conceptualization tests than did the sample of children who were not exposed to Bidirectional Translation;

(ii) for both the experimental and control groups there was a significant trend in terms of the difficulty of the different classes of verbal context - Part-Part-Whole contexts were easiest, Separating contexts were more difficult, Joining contexts were even more difficult, Equalizing contexts presented further difficulty and Comparison contexts were the most difficult;

(iii) this trend held for the four different types of conceptualization - the utilizability of addition, the utilizability of subtraction, the evocability of addition and the evocability of subtraction;

(iv) differences between addition and subtraction performance were less clear cut - in most instances performance on addition was better than performance on subtraction though this difference was not always significant;

(v) for the experimental subjects the utilizability and evocability functions were significantly associated;

(vi) within the experimental group only, there was a significant overall difference in performance;

(vii) the methodology of Bidirectional Translation would appear to significantly affect performance - reasons for this will be explored in the next chapter.

CONCLUSIONS FROM THE DATA

The research written about in this thesis was an attempt to make a contribution to the pedagogy of early number work in primary education, particularly in the areas of counting, and addition and subtraction which follow on from counting. In this, the final chapter, a number of issues need to be 'revisited' and commented upon, in an attempt to understand what was happening. But attempts at explanation imply causal mechanisms, which can be difficult to pin down, which may not be correct, and even if correct may not be complete. Nevertheless, there follows an attempt to understand the relationship between teaching methodology and test performance: firstly by the negative process of eliminating some of the likely causal factors (and acknowledging where methodological flaws preclude this!);

and

secondly by the positive process of positing a psychological concept which might substantiate the findings.

This twin-pronged approach is an attempt to leave the reader with a reasonably coherent and integrated impression of the experimental findings.

A Process of Elimination

If, in teaching young children to add and subtract, we want to promote the individual's understanding of what he/she is doing then the methodology of Bidirectional Translation would seem to be worthy of consideration for teaching purposes. Children exposed to this method perform better than children who are not. This rather sweeping generalization presupposes that the samples of control and experimental subjects were alike in all respects apart from the teaching techniques they experienced. While every reasonable sampling precaution was taken in the name of the internal and external validity factors outlined in a previous chapter, it is freely acknowledged that in the field of social science one is working with people in whom there are a multitude of extraneous variables not all of which may actually be controlled for.

One possible flaw in the design of the experiment was that control subjects were asked to comment on 'silly stories' when, in fact, they may have been confused by the referents. The word 'silly' may have implied that no real logic need be applied to the problem. Unlike the experimental subjects, the control subjects had not systematically built up an association between the term 'silly story' and the contextualization of addition and subtraction operations. With hindsight, this seems a glaring error and if it has foundation, could invalidate the data collected. Undoubtedly, the control subjects would have had experience of being told stories; if not at home, at least at school. It is also probable that they would have had some notion of what the word, silly, meant; as being slightly amusing albeit somewhat ridiculous. Although the experimental subjects did generate amusing and implausible verbal contexts, this was not the researcher's principal intention. It was her intention, merely, that the subjects should locate an addition or subtraction operation in some sort of scenario which had meaning for each child, and the use of the terms 'silly story' and 'number story' had been coined for experimental subjects to make clear the distinction between verbal context and numerical representation. If the term, 'silly story' confused some of the control subjects, which is one possible explanation to be deduced from some of the control subjects' failure to make a response to the stimulus item, it did not confuse all of the control subjects. It was not a characteristic of control subjects that they failed to respond, although many of them responded wrongly. Out of the 90 control subjects only 5 scored zero out of a total of twenty items. And while most of the control subjects performed miserably when required to evoke a concept, 85 of the 90 were able to utilize a concept, even if such utilization did not extend to all classes of verbal context. While not trying to defend what now is seen as a design fault, it would seem, on balance, that the control subjects did have some grasp of what the term, 'silly story' meant. This does not, however, preclude the possibility that the control subjects' responses might have been different if the term 'silly story' was a part of their working vocabulary, as it was for the experimental subjects. Any replications of this work would require the issue of terminology to be resolved.

Another possible factor to account for the large difference between experimental and control subjects inheres in the piloted methodology. It is possible that the **newness** of the approach, rather than the **structure** of the approach energized the participating teachers into more effective teaching. This possible source of contamination could not really be obviated in this research. An attempt to control for the Hawthorne Effect was made in having the teachers of control subjects complete a questionnaire on the teaching of addition and subtraction. The answer to the question of whether it was the newness of Bidirectional Translation or the structure of Bidirectional Translation which effected superior performance can only be found if the methodology were to be taken on board more generally by teachers and were to exposed to testing after a passage of time. In other words, it is only when Bidirectional Translation is no longer **new** that the power of its structure can be analysed. However, even at this stage it is possible to make claims for the power of Bidirectional Translation in terms of its structure. Firstly, the statistical analyses (in the previous chapter) showed that differences between the experimental and control groups were clearly greater than any variations within the experimental and control groups. This would seem to suggest that Bidirectional Translation is robust enough to withstand any differences among teachers. And secondly, participating teachers, of their own volition, made comments to the effect that the methodology assisted them insofar as:

1. it enabled the children to remain on task without prompts or reminders from the teacher;
2. the children were enthusiastic about Maths lessons, whereas in the teachers' previous experiences with the same content, Maths sessions had been a struggle for the children and tortuous for the teacher;
3. the teachers themselves had found the approach a learning experience since they had not hitherto appreciated just how much time the children needed to explore the language of addition and subtraction and unpack the meaning of the symbolism in terms of everyday events;
4. the teachers, again of their own volition, said they would use the methodology when teaching subsequent groups of young children to add and subtract.

New or old, any approach which removes pressure and 'nag' from teaching, and which gives the child 'ownership' of his/her learning, and which does not lead to the 'de-skilling' of the child is welcomed by the teacher (Ausubel & Robinson, 1969; Desforges & Cockburn, 1987; Ashman & Conway, 1989). While these claims for Bidirectional Translation cannot be fully substantiated in this piece of research, it would seem that there is at least some justification for them. As a result, the **newness** factor is not probably of much significance, insofar as the approach of Bidirectional Translation appears to support the teacher's function and the child's learning.

The final factor to be considered which might explain the significant differences between control and experimental subjects is the subjects themselves. The two groups could have been inherently so different that Bidirectional Translation neither facilitated the experimental subjects nor did alternative methodologies disadvantage the control subjects. After all, there was no pre-testing done by the researcher on any subjects, there was no standardized measure taken of their performance in number prior to the teaching of addition and subtraction. The possibility that the experimental subjects would have post-tested well irrespective of what methodology they had experienced, and the possibility that the control subjects would have post-tested badly if they had been exposed to Bidirectional Translation cannot be denied. However, it is probably unlikely that experimental and control groups each were homogeneous in all of the multitude of variables on which human beings differ. Six experimental and six control schools were involved and while traditional indicators such as intelligence and social class were not investigated, sampling procedures did attempt either to randomize or counterbalance the effects of the more obvious variables. The extent to which such attempts were successful in neutralizing intervening variables cannot really be determined. None of the participating schools were previously known to the researcher so there was no personal bias on the part of the researcher, such as selecting schools where she knew she would get entry and co-operation. But the real reason for the schools agreeing to participate may in itself have been a biassing feature. They may, for example, have agreed to participate because of some perceived increase in importance for doing so rather than for the more altruistic reasons of furthering work in children's learning. If, however, schools did agree to take part for 'the wrong reasons', this chance factor would at least apply equally to the experimental and control schools. As such, this would weaken the external validity of the experiment although not the internal validity.

The solution to the imponderables raised here would seem to lie in much larger samples being used in further replications of the work to try to negate the differences found here. If the differences between Bidirectional Translation and other methodologies still continued, then it would be reasonable to conclude that the design flaws outlined here were not critical.

A Process of Substantiation

Nevertheless, if, for the moment, it can be assumed that sampling was adequate the question that has to be asked is what was it about Bidirectional Translation that effected the difference in performance? Why did children experiencing Bidirectional Translation perform better than those children who did not?

The effects of Bidirectional Translation suggest genuine optimism in the teaching of addition and subtraction, optimism that it is possible rather than impossible to teach children to compute in ways which, from the very beginning, have face validity for them. This is not to say that teachers are redundant or that teaching addition and subtraction are trivial activities, but it does raise questions as to the traditional role of the teacher in this curricular area. The classroom folklore suggests that children come to school with virtually no numerical experience and with blank numerical minds. The classroom folklore further suggests that the remedy for this tabula rasa state is to force impressions on the minds of children by providing them with numerical experience, by showing them how to express this experience in regular mathematical forms; and that children hold such impressions in their minds by continuing to rehearse them until they 'stick'. This model clearly must have some plausibility since it has a dominant hold in the pedagogy of number. Yet the orientation of the model is at odds with the recent research which clearly demonstrates that children come to school with a wealth of numerical impressions and experience (albeit informal) which they are perfectly willing to share with teacher and peers in the school context. The teacher who fails to take account of this phenomenon is not facilitating the child's learning - the very task with which the teacher is charged!

Perhaps, as Hughes (1986) points out, the concept of **translation** is the useful thinking tool when considering the intervention the teacher should make to facilitate the child's learning:

> Mastering the formal code of arithmetic involves negotiating a complex of subtle and inter-related transitions. Some of these transitions can be distinguished: from actual to hypothetical situations, from concrete to abstract elements, from spoken to written language, from embedded to disembedded thought, from words to symbols and from the informal to the formal. This sequence is not intended to suggest any particular linear order, although clearly some transitions must precede others. At any stage in a child's mathematical development, they are therefore involved in consolidating what they already understand, and in trying to link up the novel and unfamiliar with their existing state of knowledge.

The transitions outlined above are not peculiar to mathematics, though they are an integral part of it. Such transitions can and do occur in other areas of knowledge. Thus making the transitions must be a characteristic learning tool. The idea of translation becomes a tool that the teacher can use, a tool that does not dehumanize learning because it corresponds to one aspect of the voluntary activities of mental processing. To translate **means** to turn from one form into another, to express the sense of something in an alternative form of representation. And when, as individuals we are making our translations we make them in terms of what we already **know**!

Translation then can act as a tool because it encapsulates an awareness, a precise awareness that allows the teacher to seek, and find, definite ways of helping the child to make explicit the existing 'state' of his/her own learning. Translation also makes clear to the pupils that they own a personal resource which will bring learning nearer, if they choose to use this resource. But translation is also a tool in a less metaphorical sense. As was demonstrated earlier in this thesis, translation becomes a technique for dynamically exploring the articulation of mathematical situations so that the learner becomes familiar with the interconnections of the mathematics and how the learner's own processing defines the structure.

Throughout, there has been an emphasis on translation, which may suggest that such a tool can be applied everywhere in the learning of mathematics. It is not, however, the author's intention to leave readers with the impression that translation is a multi-purpose tool capable of teaching anything. Complex learning jobs require a variety of tools, most of which have probably not been invented. The research described and discussed here has focussed on one tool that has proved useful. There is no implication that it is the only one or the only kind. Rather, the work here seems to the author to serve as a paradigm for the invention of tools to support the teaching of mathematics.

If translation is one useful tool in the teaching of addition and subtraction then perhaps another is modelling, or learning by observation. Modelling is not to be thought of as blind, mindless imitation, nor in terms of crude stimulus-response mechanisms. Rather, modelling is the adoption of selected actions and behaviours on the part of the learner, such selection being mediated by cognitive processes. What might these processes be?

1. Perception

One cannot learn much from observing another unless one attends to, or accurately perceives, the salient cues and distinctive features of the other's behaviour. In other words it is a necessary but not sufficient condition for the learner to see the model and what the model is doing. Beyond that, however, the learner must also attend to the model with enough perceptual accuracy to extract the relevant information to use in imitating the model. This is not to say that all of the modelled behaviour will be 'taken on board' by the learner. A myriad of factors involving the learner, the model and the interaction between the two can militate against the learning process.

2.Encoding

A second possible process in learning by observation concerns long-term memory storage of behaviours that have been modelled at one time or another. Crudely put, one cannot be affected much by observation of another's behaviour if one has no memory of it. Indeed without the facility to recall what the model did, the learner is unlikely to demonstrate any enduring behavioural change.

For the learner to benefit from the behaviour of the model when the model is either no longer present to serve as a guide, or present but not exhibiting the specific behaviours to be learned, the learner must code the modelled behaviour in some symbolic form which may later be recalled to enable the performance of the behaviour. This coding could be in the form of a visual image - the kind of everyday phenomenon which allows us to 'see in our mind's eye' a picture of a person or an event previously experienced. Another possible coding could be of a verbal kind - vocal or subvocal commentaries of what the model is doing which can be rehearsed internally without an overt enactment of the behaviour. Both types of coding seem intuitively plausible.

3.Recoding

A third possible process involved in learning by observation consists of translating the symbolically coded memories into appropriate behaviour. Even with accurate perception, and efficient encoding which has included rehearsal the learner may still be unable to enact the behaviour correctly. The co-ordination of component parts of the total behaviour may not be sufficiently refined. It is at this point that practice is required; practice to **learn**, not practice to the point of monotony. Furthermore, the practice will only be valuable to the learner if he/she can make self-corrective adjustments to the behaviour on the basis of informative feedback.

4. Motivation

The fourth and final process is possibly to do with positive reinforcement. Reinforcement is whatever actual or anticipated consequence of a behaviour encourages one to continue to engage in the behaviour. However well one has perceived, encoded and decoded a particular behaviour, it is unlikely to find expression in overt terms if it is negatively sanctioned, unfavourably received or in some way decidedly uncomfortable for the participant. The reinforcer can be experienced either directly or vicariously, and it can come from within the learner (i.e. the learner can find the behaviour intrinsically satisfying) or from a source external to the learner. The point of importance about reinforcement is that it provides the learner with information, information as to what to expect as a result of performing the behaviour correctly or incorrectly, and this information is essential if one is to correctly anticipate the probable consequences of one's actions and thereby make informed choices. Without the capacity for anticipatory behaviour, all manner of human thinking activity such as reasoning by analogy and hypothesis testing would be unavailable to us.

How modelling can explain Bidirectional Translation.

In Bidirectional Translation the teacher was the initial and principal model. Teachers, especially teachers of young children, are generally perceived as being of high status. Many parents will testify to their young child's constant and somewhat irritating reference to the teacher being the source of all knowledge, the fountain of all goodness and the model of excellence to which the child aspires! Beyond the general notion, however, of the teacher being a person who commands considerable attention from his/her young pupils, can Bidirectional Translation be explained in terms of modelling?

In terms of perception, there were many key points to which the learners had to attend: the connection between numbers (the abstract ideas) and the numerals (the written symbols), between words and operator signs; the connection between verbal contexts and numerical representations; the connection between verbal context, pictorial representation and numerical *representation;* the means by which answers could be found. The learner's attention to these key features was a *function of* the teacher's repeated provision of opportunity for each of these to be the focus of a learning/teaching session.

In terms of encoding, both visual imagery and verbal rehearsal were being encouraged. The learners saw, and subsequently reproduced, the construction of a numerical representation of a verbal context. They also made representations of their visual images of addition and subtraction operations when they drew pictures of their 'silly stories' (their self-generated verbal contexts). Verbal rehearsal was being made quite explicit when the learners were required to 'read' the 'number stories' and signal as to what each component part of the numerical representation meant. Verbal rehearsal was further encouraged when the learners were invited to share with their peers their methods of solution for obtaining answers.

In terms of recoding, the learners were constantly being required to effect the translation from one form to another; from verbal to numerical and from numerical to verbal. This was a critical part of Bidirectional Translation, and proportionately, took up the largest amount of teaching and learning time. At the inception of each of the steps of the methodology there were frequent false starts which needed time, patience and practice to correct.

In terms of motivation, the learners were reinforced in various ways. There was lots of praise from the teacher for even the most tentative approaches to the desired performance. As well as being directly reinforcing to the particular learner making the contribution, the praise was vicariously reinforcing to the learner's peers within the group of children who were currently with the teacher. When the learners were required to provide verbal contexts for numerical representations there was what seemed a never ending succession of offers to contribute 'silly stories'. Each learner was very willing to, and indeed clamoured to, make several contributions. This can be interpreted as a need for either direct or vicarious reinforcement from the teacher or it can be interpreted as a need for self reinforcement: the generation of yet another context by the learner being further 'evidence' for the learner that his/her little 'hypothesis' was a correct one in the given situation.

What seems to becoming clear in all of this is that modelling and translation are not discrete. Translation is a **part** of modelling, possibly in all of the cognitive processes of modelling but most certainly in the process of recoding. In the ordinary everyday world, recoding and translation are probably two referents for the same concept.

Modelling has its origins in, but is not restricted to, behaviourism, a theory which posits that all behaviour is lawfully determined, predictable and capable of being brought under environmental control through stimulus response mechanisms. The strictest form of behaviourism will not countenance mentalistic explanatory constructs on the grounds that their empirical validation is impossible. Modelling, on the other hand, while acknowledging that external stimuli and environmental responses are powerful determinants of human behaviour, gives a central role to these cognitive processes for the regulation and organization of human activity.

To the reader, it might seem that there is some tension or conflict developing. On the one hand Bidirectional Translation can be explained in terms of a psychological concept which has strong behaviourist connotations. On the other hand, however, the researcher's whole thesis was driven by the premise that the child's **conceptualization** and **understanding** of what he/she was doing was of primary consideration, and such conceptualization was held to be rooted in Piagetian ideas. So the question which arises from this is, to what extent is modelling compatible with Piagetian or neo-Piagetian thinking where concern for individual development, discovery learning, and active involvement on the part of the child are all of paramount importance?

Fifteen or twenty years ago, when the implications of Piagetian theory were becoming better understood, a **psychological** explanation for the paucity of achievement in number work at school would have been that children were being asked to engage with material for which they had not yet developed the relevant schemata. Since, according to Piaget (1964), "learning is subordinate to development", Piagetian theory was commonly interpreted as meaning that there are definite limits on the extent to which the child's progress can be accelerated by environmental influences, and that the passage of time rather than experience promotes internal growth. Whatever misunderstandings there may have been in this explanation, a colloquial description for this state of affairs was that 'the child was not ready'. The notion of readiness seemed to be subject to various interpretations. At one extreme, it became a universal but tautologous explanation: whenever a child failed to learn, or refused to engage in a task, the child must not have been 'ready' (Sharp & Green, 1975). More usually, 'readiness' would be strongly associated with biological maturation, because Piaget had shown that children's thinking was different for different age groups. This can be criticised as an unjustified inference of causation.

While some sort of readiness notion appears to be implicit in Piagetian theory, it nevertheless does not constitute justification for delaying or terminating attempts to teach the child whatever he/she is allegedly not ready for! Taken to its logical conclusion, one could wait forever and some children would still not be ready. The notion of readiness might be a more useful one if it were stripped of the underlying assumption of biological maturation and recast in terms of conceptual maturation. In order to engage in a specific type of complex cognitive task (such as mathematics or reading, which occupies a large part of early schooling), the child must have a variety of abilities and predispositions, which can be identified if we, as teachers, understand the task and the demand it poses. To be able to learn to add and subtract involves knowing, at least, that these are activities which, in the real world, people engage in frequently in a wide variety of contexts. It further involves learning how to engage in these activities and, where need be, how to communicate the findings of these activities in recognized and recognizable forms. By acknowledging that each of the component parts is necessary and by structuring these component parts of addition and subtraction into a form which progressively embraces all of the parts (which was what the methodology of Bidirectional Translation was attempting to do), the need to make assumptions about the child's state of readiness is obviated. Specific learning experiences 'prepare' the child for subsequent learning. Furthermore, a structure such as Bidirectional Translation provides a practical procedure for instituting remedial intervention.

The 'mystique' of discovery learning (as being distinct from, and superior to, reception learning) can similarly be dismantled. There seems to have developed in education, the polarised view that only discovery learning is meaningful and that all reception learning is rote in nature. Discovery learning is claimed by its advocates (Shulman & Keisler, 1966) to be inherently more meaningful, to be retained longer and to motivate further learning more effectively than reception-learning approaches. But sight seems to have been lost of what Ausubel and others (Ausubel & Robinson, 1969) were saying which was that if the learner attempts to retain an idea by relating it to what he/she already knows, and thereby make some sense of it, then meaningful learning will result. By contrast, if the learner merely memorizes an idea, without relating it to his/her existing knowledge, then rote learning is said to take place.

In what Ausubel was saying there is nothing to suggest that the only meaningful learning which takes place is effected through 'discovery'. For meaningful learning to take place it is critical that what is to be learned should somehow connect with what is already known. Sometimes this connection is best made by the teacher structuring the material which is to be learned and if all that is required on the part of the learner is assimilation, then the teacher's job is relatively easy. If, however, accommodation is required of the learner the teacher's job is more complex in that two possibilities emerge. One is to leave the learner to restructure his/her schemata, as in some form of discovery learning. The other is to find out more about these schemata and shape the teaching accordingly: a technique such as as Bidirectional Translation can help in that the structured dialogue which forms the spine of the approach allows the learner to reveal his/her existing knowledge.

A main disadvantage of discovery learning is that there must be a high prospect of success to sustain the learner through the process of trial and error. Such 'discovery' as there is, is likely to be time consuming and not easy to guarantee. Of perhaps lesser importance in practical terms is the objection to discovery learning raised by Ausubel & Robinson (1969). They point out that it is a repudiation of the very concept of culture. We do not make progress by continually 're-inventing the wheel'. Modelling, on the other hand, allows the learner to observe the target response which, if complex, might never be 'discovered'.

Active involvement, the third of the so called Piagetian ideas, is similarly vague. The assumption that has been espoused by many educators, particularly those who claim to be 'progressive' in their philosophy, is that children need to learn by doing. Few psychologists would discount the importance of direct personal experience on a task in assisting learning. It is, however, quite another thing to insist that a child must be engaged in direct performance before any real learning can occur. Consider the practical problem in requiring children to directly participate: the teacher is faced with the child who is reluctant to perform on a particular task, particularly if the task is unfamiliar. Even the promise of attractive rewards for task performance does not encourage or effect 'activity'. An intuitive explanation for this phenomenon is that the child is afraid of failure. Whatever the real explanation is, if inducements do not succeed, the teacher is left with three alternatives: one is to 'force' the child to participate, which is somewhat self-defeating and objectionable on moral grounds; another is to postpone the teaching until some future time when the child is 'ready', which, as has been argued above, is something of a fallacy. The third alternative (and for the author, the only sensible course of action) is to have the 'recalcitrant' child watch another child/other children participate first. In most situations, the observing child will, after a few demonstrations, 'assert' his/her right to participate: yet another example of the child learning through modelling.

At this juncture it is perhaps worth making clear what Piaget himself regarded as active learning. As Schwebel & Raph (1973) point out the nature of the activity is critical. That "children move about the classroom, that seats are not fixed, and that children even hop, skip, and jump do not make the active process educative". What makes any active process educative is the effect of the experience on the child's subsequent behaviour. If the activity causes no intellectual change, then the activity has not facilitated learning. Kamii (1973), in summarizing what Piaget said about active methods, points out that what makes an 'active' method active is not the external actions of the learner but the criterion of the learner actively constructing his/her own knowledge. This in turn requires some systematization on the part of the teacher such that "structuring, elaborating and reasoning processes" are a genuine part of the contact between teacher and learner.

However, along with the more common misrepresentations of Piagetian theory, there is perhaps a small but nevertheless significant gap in the folklore about what Piaget really said. And that is that he himself recognized modelling to be germane to intellectual development. Around the time when the young child is making the transitions from the sensori-motor to the pre-operational stages of development, the child can be seen employing what is referred to in Piagetian terms as deferred imitation. As Flavell (1977) reports:

> One of Piaget's children, for example, watched in mute fascination while another child threw a three-star temper tantrum. She then produced an excellent imitation of it the next day.

For the observed behaviour to be reproduced so accurately after a temporal delay means that in observing the other, the child must have represented the event mentally; a classic example of modelling.

Nor is deferred imitation only to be witnessed at the start of the pre-operational stage. The teacher of young children frequently has opportunity to witness his/her pupils in 'free' or sociodramatic play. Here the teacher sees the child being mummy or daddy or whoever. The enactment of such roles involves dialogue and behaviours which, without a memory of having observed a related or similar scenario, would be impossible for the young child to produce. Clearly then, modelling is a powerful learning medium for young children. They would appear to use it spontaneously. That being the case, it makes sense for the teacher to capitalize on it.

For the teacher who has professed but ill-formed notions of 'child-centred' education, the idea of teaching through explicit modelling may seem an anathema. But child-centred education need not imply non-intervention while we wait for growth. Rather it means active partnership between teacher and child ensuring that children's mental experiences are structured in ways that are most likely to be fruitful. For this to happen it seems important that the child should understand the purpose of the activity in which he/she is engaging, should anticipate success, and should be engaged in the task to the extent that he/she can direct his/her full attention to the learning that is supposed to be brought about by the task. This thesis has been but a tiny attempt in this direction. Hopefully it will encourage a more eclectic approach in the teaching of early number work.

APPENDIX 1

Short description of Logic People.

Logic People are a variation of the Logic Blocks which are to be found in many infant departments. Sometimes the Logic Blocks are referred to as Attribute Blocks because each set consists of blocks of plastic, not all of which have the same attributes. Typically a set of Logic Blocks (or Attribute Blocks) consists of
triangles, squares, circles and rectangles which are:
red, blue or yellow;
large or small;
thick or thin.

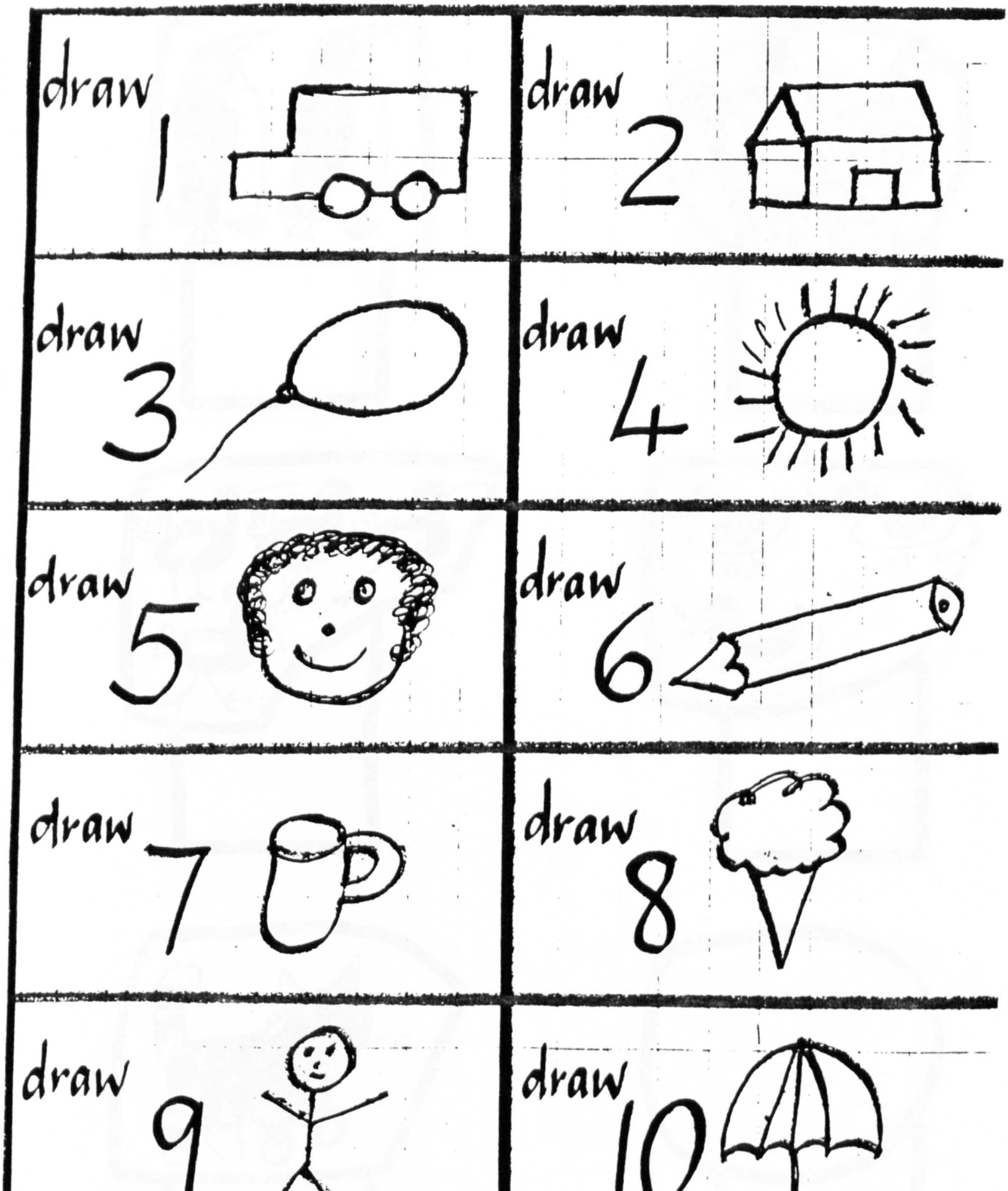
By playing different games with this material, the child has the experience of classifying the blocks of plastic according to differing attributes.

Logic People are less abstract than the Logic Blocks in that the pieces of plastic are people rather than geometric shapes. A set of Logic People consists of
men, women, girls and boys who are:
red, blue, yellow or green and who are
sitting, walking or standing.

There was nothing inherent in the Logic People which caused the researcher to use them as countables. They were used because they were easy for the children to lift, hold and move around, and because the children were already familiar with them through free play activities.

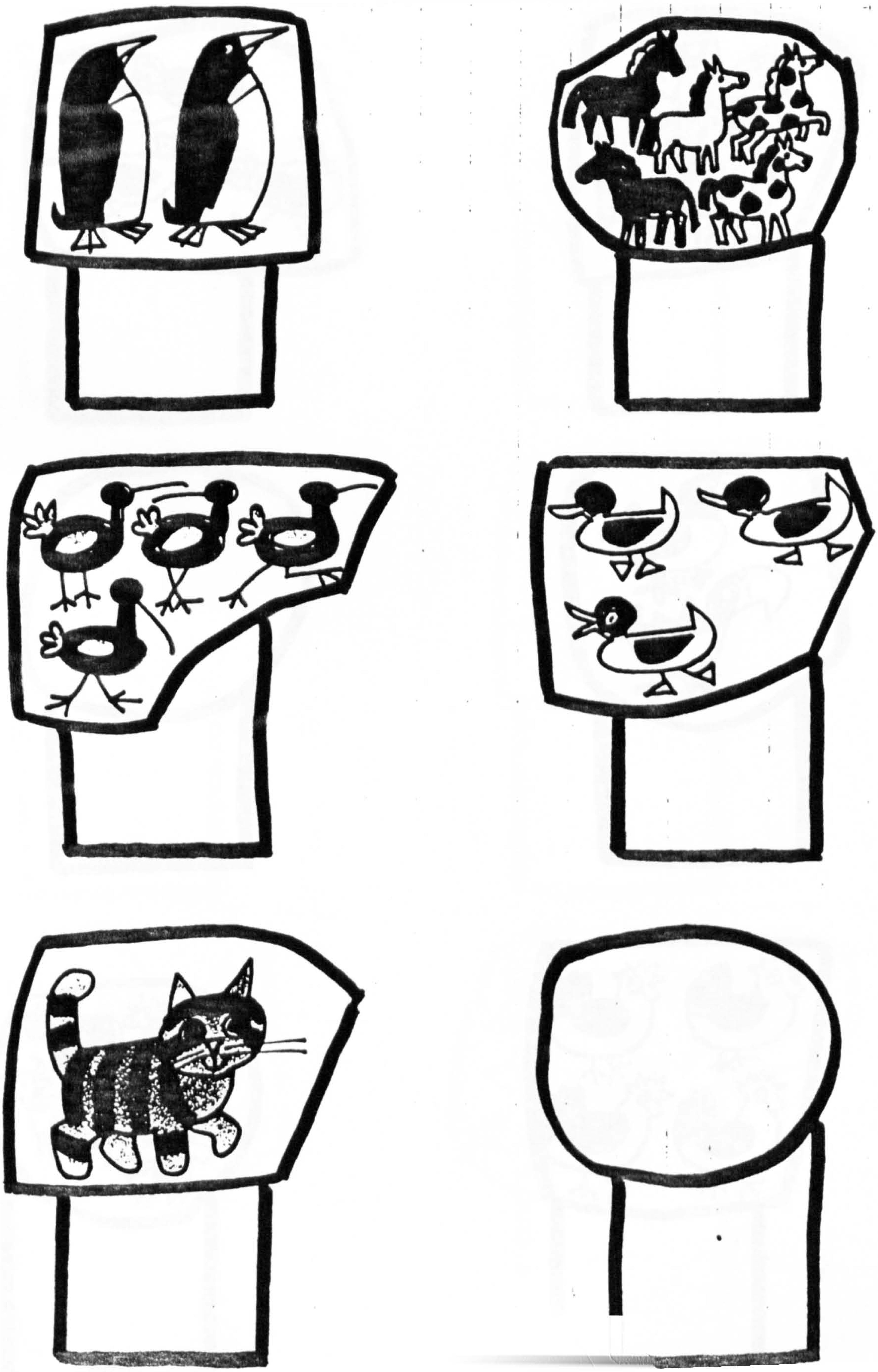
APPENDIX 2

Stimulus cards for Representing Given Numerosities.



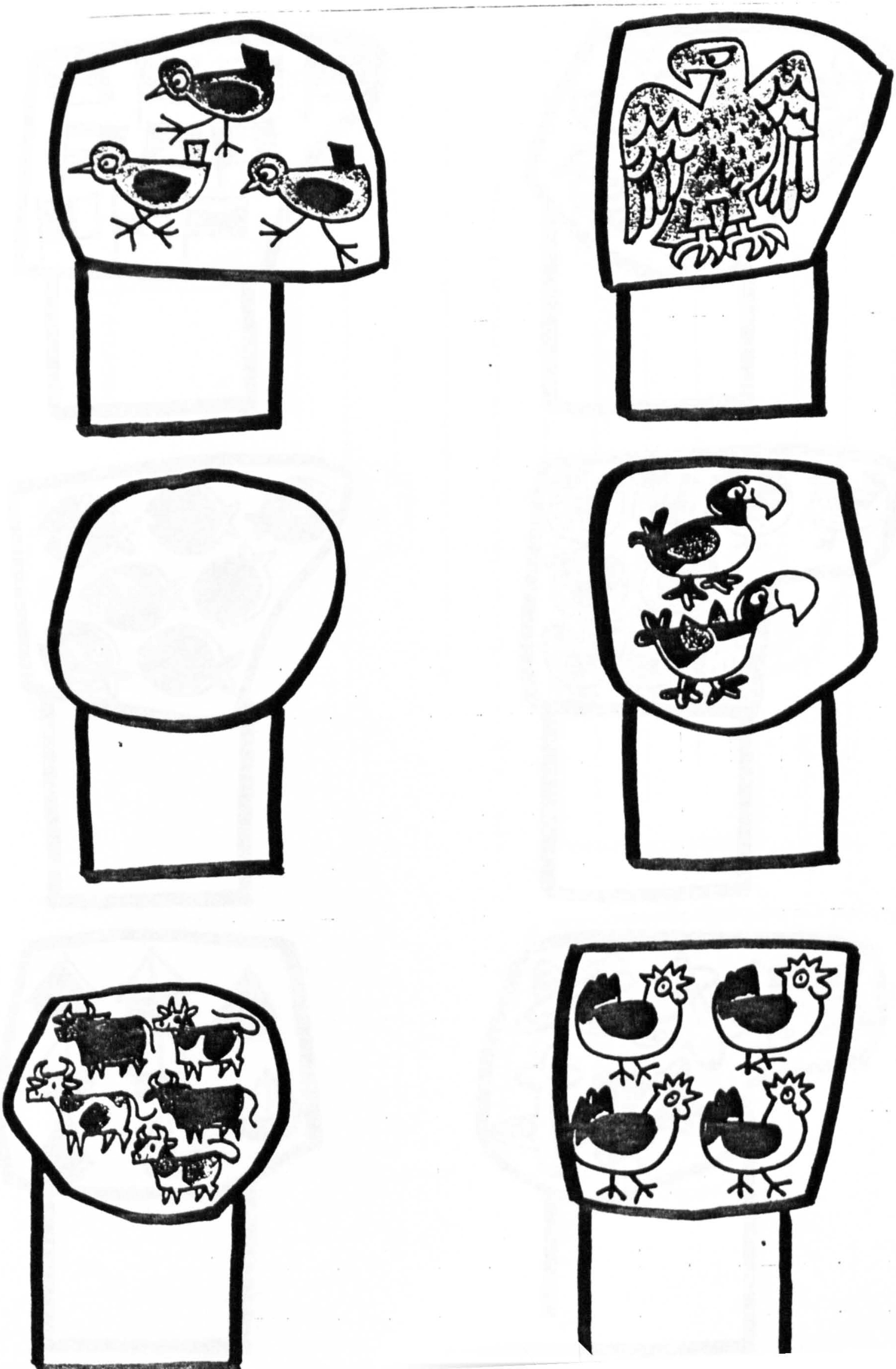
APPENDIX 3

Stimulus material for Representing Obtained Numerosities.



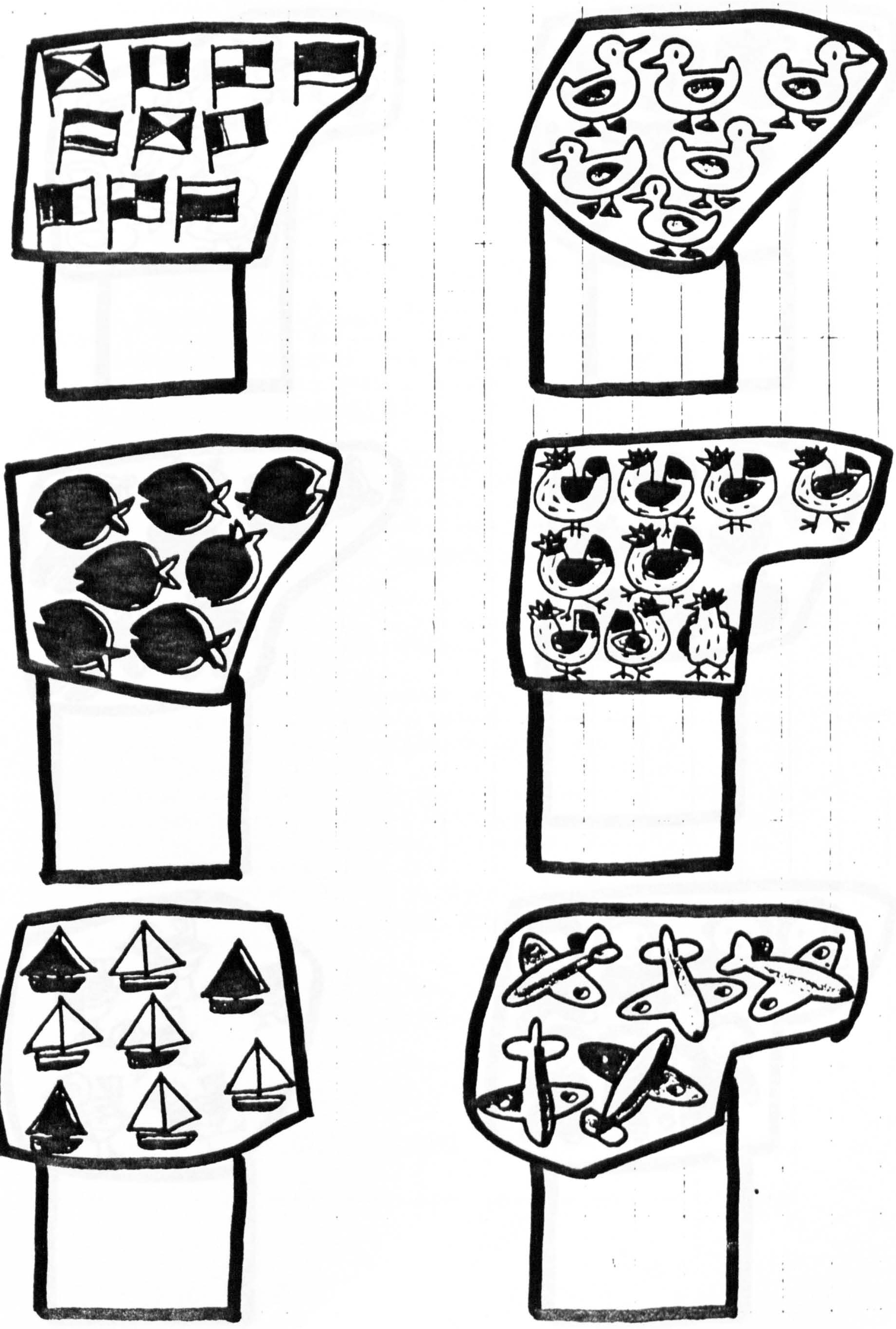
APPENDIX 3

Stimulus material for Representing Obtained Numerosities.



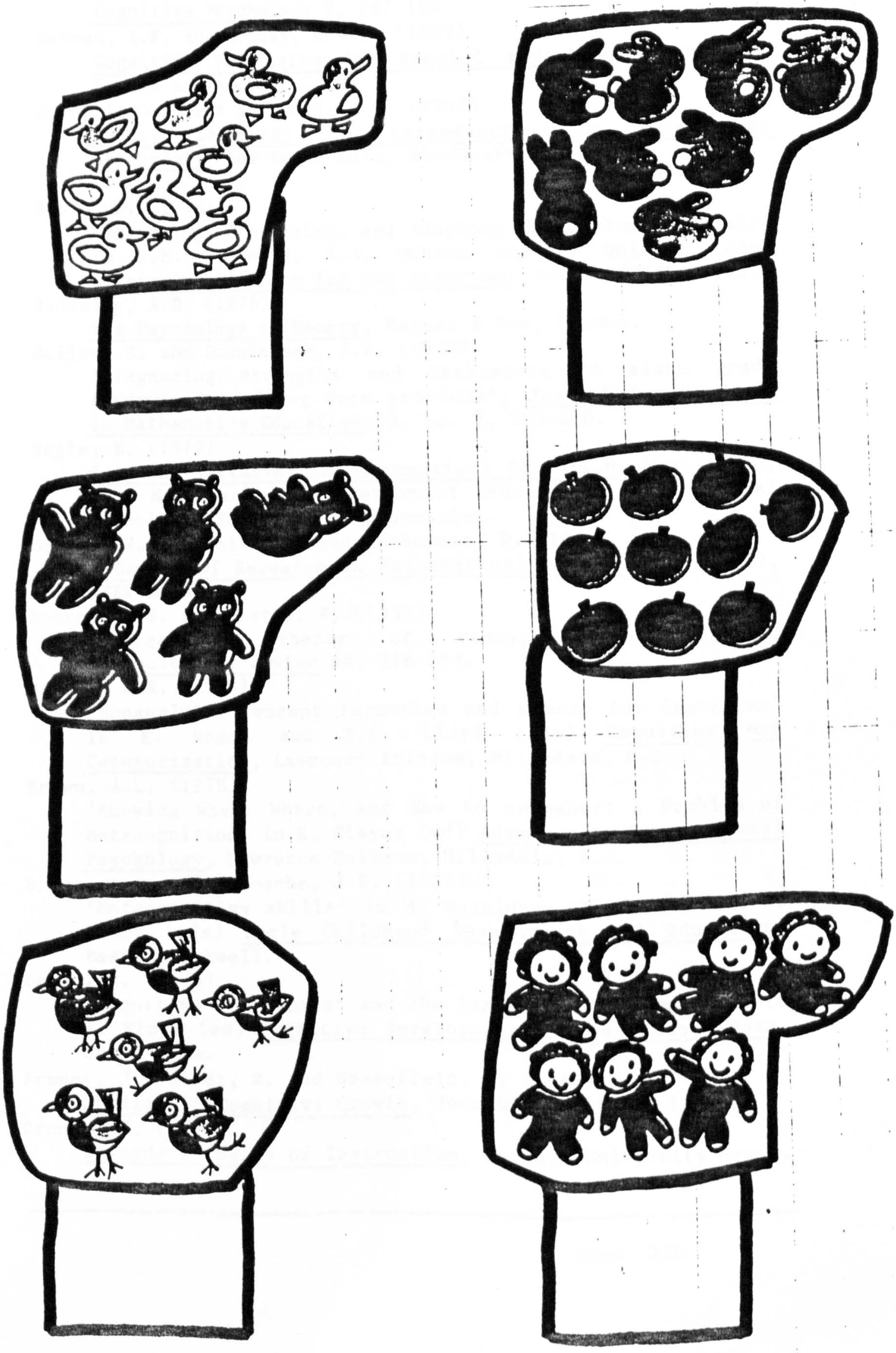
APPENDIX 3

Stimulus material for Representing Obtained Numerosities.



APPENDIX 3

Stimulus material for Representing Obtained Numerosities.



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