THE SECOND DUAL OF A BANACH ALGEBRA

by

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CONTENTS

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DECLARATION

I hereby declare that this thesis has been composed by myself, that the work of which it is a record has been done by myself (unless indicated otherwise), and that it has not been accepted in any previous application for a higher degree.

INTRODUCTION

Let A be a Banach algebra over a field $\mathbb{F}$ that is either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$, and let A' be its first dual space and A'' its second dual space. R. Arens in 1950 [2], [3], gave a way of defining two Banach algebra products on A'', such that each of these products is an extension of the original product of A when A is naturally embedded in A''. These two products may or may not coincide. Arens calls the multiplication in A regular provided these two products in A'' coincide.

Perhaps the first important result on the Arens second dual, due essentially to Shermann [17] and Takeda [18], is that any C*-algebra is Arens regular and the second dual is again a C*-algebra. Indeed if A is identified with its universal representation then A'' may be identified with the weak operator closure of $\hat{A}$.

In a significant paper Civin and Yood [7], obtain a variety of results. They show in particular that for a locally compact Abelian group G, $L^1(G)$ is Arens regular if and only if G is finite. (Young [24] showed that this last result holds for arbitrary locally compact groups.) Civin and Yood also identify certain quotient algebras of $[L^1(G)]''$.

Pak-Ken Wong [22] proves that $\hat{A}$ is an ideal in A'' when A is a semi-simple annihilator algebra, and this topic has been taken up by S. Watanabe [20], [21] to show that $[L^1(G)]^\wedge$ is ideal in $[L^1(G)]''$ if and only if G is compact and $[M(G)]^\wedge$ is an ideal in $[M(G)]''$ if and only if G is finite. One should also note in this context the well known fact that if E is a reflexive Banach space with the approximation property and A is the algebra of compact operators on E, (in particular A is semi-simple annihilator algebra) then A'' may be identified with $BL(E)$.

S.J. Pym [The convolution of functionals on spaces of bounded functions, Proc. London Math. Soc., (3) 15 (1965)] has proved that A is Arens regular if and only if every linear functional on A is weakly almost periodic. A general study of those Banach algebras which are Arens regular has been done by N.J. Young [23] and Craw and Young [8].

But in general, results and theorems about the representations of A'' are rather few.

In Chapter One we investigate some relationships between the Banach algebra A and its second dual space. We also show that if A'' is a C*-algebra, then $*$ is invariant on A.

In Chapter Two we analyse the relations between certain weakly compact and compact linear operators on a Banach algebra A, associated with the two Arens products defined on A''. We clarify and extend some known results and give various illustrative examples.

Chapter Three is concerned with the second dual of annihilator algebras. We prove in particular that the second dual of a semi-simple annihilator algebra is an annihilator algebra if and only if A is reflexive. We also describe in detail the second dual of various classes of semi-simple annihilator algebras.

In Chapter Four, we particularize some of the problems in

Chapters Two and Three to the Banach algebra $\ell^1(S)$ when S is a semigroup. We also investigate some examples of $\ell^1(S)$ in relation to Arens regularity.

Throughout we shall assume familiarity with standard Banach algebra ideas; where no definition is given in the thesis we intend the definition to be as in Bonsall and Duncan [6]. Whenever possible we also use their notation.

CHAPTER 1

Let A be a Banach algebra (over the real or complex field). Let A' and A'' denote the first and second dual spaces of A. Let $a, b, \ldots$ denote elements of A ; $f, g, \ldots$ denote elements of A' ; $F, G, \ldots$ denote elements of A''.

For each $f \in A'$, $a \in A$ we define $fa \in A'$ by the rule:

$$fa(b) = f(ab) \qquad b \in A .$$

For each $F \in A''$, $f \in A'$ we define $Ff \in A'$ by the rule:

$$Ff(a) = F(fa) \qquad a \in A .$$

For each pair of $F, G \in A''$, we define $FG \in A''$ by the rule:

$$FG(f) = F(Gf) \qquad f \in A' .$$

These definitions were introduced by Arens [2], [3] who showed the definition of FG as a product of F and G yields an associative multiplication on A'' which makes A'' into a Banach algebra. Throughout we call this multiplication in A'', the first Arens product. The natural embedding of A into A'' will be denoted by $\hat{A}$. As noted by Arens [2], the natural embedding is an isometric isomorphism when A'' is considered as a Banach algebra under the first Arens product.

Arens [3] has considered also the following multiplication in A''.

For each $f \in A'$, $a \in A$ define $af \in A'$ by the rule:

$$af(b) = f(ba) \qquad b \in A .$$

For each $F \in A''$, $f \in A'$ define $fF \in A'$ by the rule:

$$fF(a) = F(af) \qquad a \in A .$$

Finally, for $F \in A''$, $G \in A''$ define $F\#G$ by the rule:

$$F\#G(f) = G(fF) \qquad f \in A' .$$

Again the definition of $F\#G$ as product makes A'' into a Banach algebra. We call this multiplication in A'' the second Arens product.

1.1 Definition We call A Arens regular provided $FG = F\#G$ for all $F, G \in A''$.

As was noted in [3] the multiplication FG is w*-continuous in F for fixed $G \in A''$ and $F\#G$ is w*-continuous in G for fixed $F \in A''$. Also $\hat{x}G = \hat{x}\#G$ is w*-continuous in G for fixed $x \in A$. The multiplication in A is regular if and only if FG is also w*-continuous in G for fixed F , or $F\#G$ is w*-continuous in F for fixed G .

Clearly if A is commutative, A'' is commutative if and only if A is Arens regular.

1.2 Proposition. If A is commutative, then $FF = F\#F$ for every $F \in A''$.

Proof. We have

$$ab = ba, \qquad a, b \in A .$$

$$fa(b) = f(ab) = f(ba) = af(b), \qquad f \in A';\ a, b \in A .$$

$$fa = af, \qquad f \in A';\ a \in A .$$

$$Ff(a) = F(fa) = F(af) = fF(a), \qquad F \in A'';\ f \in A';\ a \in A .$$

$$Ff = fF, \qquad F \in A'';\ f \in A' .$$

$$FF(f) = F(Ff) = F(fF) = F\#F(f), \qquad F \in A'';\ f \in A' .$$

$$FF = F\#F \qquad F \in A' . \quad \Delta$$

Notation. For a subspace J of a Banach space A, we define:

$$J^{\perp} = \{f \in A' : f(a) = 0,\ a \in J\}.$$

Let A be a commutative Banach algebra, M the closed linear subspace of A' spanned by the multiplicative linear functionals on A. Then by II - 4-18-a [10], $M' \approx A''/_{M^{\perp}}$, and by Theorem 3.7.[7] $A''/_{M^{\perp}}$ is semi-simple and commutative. Also by Lemma 3.16 [7] the mapping $T : A \to A''/_{M^{\perp}}$ defined by:

$$T(a) = \hat{a} + M^{\perp} \qquad a \in A$$

is a continuous homomorphism. Now $a \in \ker(T)$ if and only if $\hat{a} \in M^{\perp}$, i.e. $\phi(a) = 0$ for every $\phi \in M^{\perp}$, i.e. $a \in \mathrm{rad}(A)$. We summarise these remarks in:

1.3 Proposition. Let A be a commutative Banach algebra, M the closed linear subspace of A' spanned by the multiplicative linear functionals on A, and let M' have the multiplication induced by the isomorphism $M' \approx A''/M^{\perp}$. Then there exists a continuous homomorphism $T : A \to M'$ with kernel $\mathrm{rad}\, A$, and M' is semi-simple and commutative.

1.4 Proposition. Let A be a commutative Banach algebra, M the closed linear subspace of A' spanned by the multiplicative linear functionals on A. Let $B = A''/_{M^{\perp}}$ and let N be the closed linear subspace of B' spanned by the multiplicative linear functionals on B. Then there exists a continuous and 1-1 linear mapping of M into N.

Proof. Let f be a multiplicative linear functional on A. Then by Lemma 3.6.[7] $\hat{f}$ is a multiplicative linear functional on A''. Since $\hat{M}(M^{\perp}) = 0$ we may define $T : M \to N$ by:

$$T\phi([F]) = \hat{\phi}(G) \qquad G \in [F].$$

If $T\phi_1 = T\phi_2$. Then $\hat{\phi}_1 = \hat{\phi}_2$, $\phi_1 = \phi_2$. Therefore T is 1-1. Evidently T is norm decreasing. Δ

1.5 Examples. (i) Let $A = \ell^1$, the algebra of absolutely convergent series of complex numbers, with the usual norm, and let the multiplication in A be defined co-ordinatewise. Then by Theorem 4.2.[7], $A'' = \hat{A} \oplus M^\perp$. So $A \approx B$ and $M \approx N$.

(ii) Let G be a locally compact infinite abelian group, and let $A = L^1(G)$ the group algebra of G. Then by Theorem 3.17 [7], $B = A''/_{M^\perp}$ is isometrically isomorphic to the algebra of all regular Borel measures on the almost periodic compactification of G, with multiplication taken as convolution. So $B \not\approx A$ and we can get a continuous embedding of M into N.

1.6 Proposition. Let A be commutative and let A'' have identity E for one of the Arens products. Then E is the identity element for the other product.

Proof. For $F \in A''$, $FE = EF = F$. Then for every $f \in A'$ we have:

$$F\#E(f) = E(fF) = E(Ff) = EF(f) = F(f) .$$

Therefore $F\#E = F$. Similarly $E\#F = F$. Also by similar way we can get $FE = EF = F$ if $F\#E = E\#F = F$. Δ

In fact, in the above case, left identity for one product is the right identity for the other one, and right identity for one product is the left identity for the other product.

1.7 Definition. A left approximate identity for A is a net $\{e_\lambda\}$ in A such that:

$$e_\lambda x \to x \qquad x \in A \qquad (1)$$

A bounded left approximate identity is a left approximate identity which is also a bounded net. Right approximate identities are similarly defined by replacing $e_\lambda x$ in (1) by xe_λ. A two-sided approximate identity is a net which is both a left and a right approximate identity.

By Proposition 28.7 [6], A'' with respect to the first Arens product has a right identity if and only if A has a bounded right approximate identity. By similar proof we have

__1.7 Proposition.__ The Banach algebra A'' with respect to the second Arens product has a left identity if and only if A has a bounded left approximate identity.

Since $\hat{a}f = af$ and $f\hat{a} = fa$ for every $a \in A$ and $f \in A'$, we get $AA' \subset A''A'$ and $A'A \subset A'A''$. Next we show that if A has a bounded two-sided approximate identity, then $A''A' = A'A'' = A'$ and we give an example which has bounded two-sided approximate identity but $A'A \neq A'$.

__1.8 Proposition.__ If A has a bounded right approximate identity, then $A''A' = A'$. If A has a bounded left approximate identity, then $A'A'' = A'$.

__Proof.__ Let $\{e_\lambda\}$ be a bounded right approximate identity in A. Then by Proposition 28.7 [6], A'' has a right identity E. So $A''A' = A'$. If $\{e_\lambda\}$ is a bounded left approximate identity in A, then it has a weak* cluster point $E \in A''$. Now for every $f \in A'$, $a \in A$ we have:

$$\hat{e}_\lambda(af) = af(e_\lambda) = f(e_\lambda a) \to f(a) .$$

Therefore:

$$fE(a) = E(af) = f(a), \qquad fE = f \ .$$

So $A'A'' = A'$. Δ

Note that by Corollary 28.8 [6], a weak* cluster point E of a bounded left approximate identity $\{e_\lambda\} \subset A$ is a left identity in A'' , if A is Arens regular.

<u>1.9 Proposition.</u> There exists a semi-simple commutative annihilator algebra A with bounded two-sided approximate identity such that $A'A \neq A'$.

<u>Proof.</u> Let $A = L^1(G)$ the group algebra of a compact abelian group G . Then by A.3.1 [15], A is semi-simple with bounded two-sided approximate identity, and by remark page 182 [6], A is a dual algebra. Now suppose that $A'A = A'$. In Chapter 3 we show that if A is a semi-simple annihilator algebra with $A'A$ dense in A' , then:

$$A'' = \hat{A} \oplus \operatorname{ran}(A'') \ .$$

So $A''/_{\operatorname{ran}(A'')} = \hat{A}$.

But, by Theorem 3.17 [6], $A''/_{\operatorname{ran}(A'')} \approx M(G)$ the algebra of all regular Borel measures on the almost periodic compactification of G with multiplication taken as convolution. Δ

In attempting to obtain some stronger results involving approximate identities, one is led to the following definition.

<u>1.10 Definition.</u> $\{e_\lambda\}$ is a bounded uniform left approximate identity if for every $a \in A$, $e_\lambda a \to a$ uniformly on the unit sphere of A .

However, as shown by P.G. Dixon, the above definition is simply equivalent to having a left identity.

1.11 Proposition. (P.G. Dixon). Let A be a Banach algebra and let $e \in A$ be such that, for some α, $0 < \alpha < 1$,

$$\|ex - x\| \leq \alpha \|x\| \qquad x \in A .$$

Then A has a left identity element.

Proof. Let $T_e \in BL(A)$ be defined by:

$$T_e x = ex \qquad x \in A .$$

Then $\|T_e - I\| \leq \alpha < 1$.

So T_e is invertible, and

$$T_e^{-1} = (I - (I - T_e))^{-1} = I + (I - T_e) + (I - T_e)^2 + \ldots$$

Let $u = T_e^{-1} e \in A$. Then:

$$(T_e^{-1} e)x = [(I + (I - T_e) + (I - T_e)^2 + \ldots)e]x$$

$$= [e + (I - T_e)e + (I - T_e)^2 e + \ldots]x$$

$$= ex + (I - T_e)ex + (I - T_e)^2 ex + \ldots$$

$$= (I + (I - T_e) + (I - T_e)^2 + \ldots)ex$$

$$= (T_e^{-1})(ex) = (T_e^{-1})(T_e x) = x , \implies ux - x = 0 . \qquad \Delta$$

A bounded uniform right approximate identity is similarly defined by replacing $e_\lambda x$ in 10 by xe_λ. Again by similar argument, if A has a bounded uniform right approximate identity, then A has a right identity.

By 9.13 iv [6] A' is a Banach right A-module under:

$$fa(x) = f(ax) \qquad f \in A', \ x \in A .$$

A' is a Banach left A-module under:

$$af(x) = f(xa) \qquad f \in A', \ x \in A .$$

And A' is a Banach A-bimodule under fa and af as module multiplications. Also by 9.13 V [6] A' is a Banach left A''-module under Ff as a module multiplication, when A'' has the first Arens product, and A' is a Banach right A'-module under fF as a module multiplication, when A'' has the second Arens product. It is a routine matter to verify that A' is a Banach A''-bimodule under fF and Ff as module multiplications if A'' is commutative and A has identity element.

<u>1.12 Proposition.</u> If $\{e_\lambda\}$ is a bounded right approximate identity for A, then:

$$\{fa : f \in A',\ a \in A\} = \{g \in A' : \|ge_\lambda - g\| \to 0\} .$$

<u>Proof.</u> Let $g \in A'$ and $g = fa$ for some $f \in A'$ and $a \in A$. Then:

$$\|ge_\lambda - g\| = \|fa\, e_\lambda - fa\| = \|f\, ae_\lambda - fa\|$$

$$= \|f(ae_\lambda - a)\| \le \|f\|\ \|ae_\lambda - a\| \to 0 .$$

Conversely, since A' is a right A-module under module multiplication fa $(f \in A',\ a \in A)$, and A has bounded right approximate identity, by Theorem 32.22 [13], $A'A$ is closed in A'. Δ

<u>1.13 Lemma.</u> Let A be a Banach algebra, and B be a left (right) Banach A-module. Let $\{e_\lambda\}$ be a bounded left (right) approximate identity in A. Then $AB = B$ $(BA = B)$ if and only if $\{e_\lambda\}$ is a left (right) approximate identity for B.

<u>Proof.</u> Let $AB = B$, and let $b \in B$. Then we have to prove:

$$e_\lambda b - b \to 0 .$$

But we have $b = ac$ for some $a \in A$ and $c \in B$. Therefore

$$\|e_\lambda b - b\| = \|e_\lambda (ac) - ac\| = \|(e_\lambda a)c - ac\|$$

$$= \|(e_\lambda a - a)c\| \leq K\|e_\lambda a - a\| \to 0 .$$

Conversely, by Theorem 11.10 [6], we get $AB = B$. Similarly we can prove $BA = B$ if and only if $\{e_\lambda\}$ is a bounded right approximate identity for B.

Now, let A' be a Banach right A''-module under Ff. Then:

$$
\begin{array}{lll}
 & G\,Ff = FG\,f & F, G \in A''; f \in A' . \\
\Longleftrightarrow & GF\,f = FG\,f & F, G \in A''; f \in A' . \\
\Longleftrightarrow & GF\,f(a) = FG\,f(a) & F, G \in A''; f \in A'; a \in A . \\
\Longleftrightarrow & GF(fa) = FG(fa) & F, G \in A''; f \in A'; a \in A .
\end{array}
$$

This gives A'' commutative provided $\{fa : f \in A', a \in A\}$ is dense in A'. This is certainly true if A has a right unit, or by Lemma 13, if A has a bounded right approximate identity for the right module A'. Similar result can be obtained when A' is a left A''-module under fF.

<u>1.14 Corollary.</u> Let $\{e_\lambda\}$ be a bounded left (right) approximate identity for A. Then $AA' = A'$ $(A'A = A')$ if and only if $\{e_\lambda\}$ is a left (right) approximate identity for A'.

<u>Proof.</u> Since A' is a left Banach A-module under module multiplication af and a right Banach A-module under module multiplication fa, Lemma 13 gives the proof. Δ

<u>1.15 Proposition.</u> The Banach left A''-module A' is faithful if A has a unit.

Proof. Let $f \in A'$ and $Ff = 0$ for every $F \in A''$. Then

$$Ff(a) = F(fa) = 0 \qquad a \in A$$

So: $$fa = 0, \qquad a \in A$$

$$fa(1) = f(a) = 0 \qquad a \in A$$

So: $$f = 0 . \quad \Delta$$

In proposition 15, in fact, it is sufficient to have A^2 dense in A.

Let $a \in A$. Define the map B_a on A' by:

$$B_a f = fa \qquad f \in A' .$$

For $F \in A''$ let $\pi(F)$ be the map on A' defined by:

$$\pi(F)f = Ff \qquad f \in A'$$

Let $C = \text{com}\{B_a : a \in A\} = \{T \in BL(A') : TB_a = B_a T\}$.

1.16 Theorem. If A has a unit, then $\pi : A'' \to C$ is a bicontinuous isomorphism. and if A is unital, then π is an isometry.

Proof. Let $F \in A''$, $a \in A$. Then, since $Ff\, a = F\, fa$ $(f \in A')$, we have:

$$B_a \pi(F)\ f = B_a(\pi(F)f) = Ff\ a = F\ fa = F\ B_a f$$

$$= \pi(F)\ B_a f = \pi(F) B_a\ f .$$

Therefore:

$$B_a \pi(F) = \pi(F) B_a .$$

Given $\phi \in C$, define $F(f) = \phi f(1)$. Then $F \in A''$ and for every $a \in A$, we have:

$$\pi(F)f\ (a) = Ff(a) = F(fa) = \phi\ fa(1) = \phi\ B_a f(1)$$

$$= B_a \phi\ f(1) = B_a(\phi f)\ (1) = (\phi f)a\ (1) = \phi f(a) .$$

Therefore:

$$\pi(F)f = \phi f. \quad \text{i.e.} \quad \pi \text{ is onto.}$$

Clearly π is linear and one-one. Now for every $F, G \in A''$, $f \in A'$ and $a \in A$, we have:

$$\pi(FG)f\ (a) = (FG)f\ (a) = FG(fa) = F(G\ fa)$$

$$= F(Gf\ a) = F(\pi(G)f\ a) = F\ \pi(G)f(a) = \pi(F)\pi(G)f\ (a)\ .$$

Therefore:

$$\pi(FG) = \pi(F)\ \pi(G)\ .$$

Also for $F \in A''$, since A' is a Banach right A-module under fa, we have:

$$\|\pi(F)\| = \sup_{\|f\|\le 1} \|Ff\| = \sup_{\|f\|\le 1}\ \sup_{\|a\|\le 1} |F(fa)|$$

$$\le \sup_{\|f\|\le 1}\ \sup_{\|a\|\le 1} K\|F\|\ \|f\|\ \|a\| = K\|F\|,$$

for some positive K. Therefore π is continuous, and Banach isomorphism Theorem gives that π is bicontinuous.

Now let A be unital. Then for every $F \in A''$;

$$\|\pi(F)\| = \sup_{\|f\|\le 1} \|Ff\| = \sup_{\|f\|\le 1}\ \sup_{\|a\|\le 1} |F(fa)|$$

Since $f1 = f$ and $\|1\| = 1$, we have:

$$\|\pi(F)\| \ge \sup_{\|f\|\le 1} |F(f)| = \|F\|. \qquad \Delta$$

<u>1.17 Corollary</u>. If A is finitely generated, then A'' may be identified with the commutant of a finite set of operators. For example, if $A = \ell^1(\mathbb{Z})$. Then A'' can be identified by commutant of the bilateral shift on $\ell^\infty(\mathbb{Z})$. If $A = \ell^1(FS(2))$, where

FS(2) is free semigroup on two symbols, then A'' is isometric with the commutant of B_u and B_v, where u and v are the generators of FS(2) .

Sherman [17], Takeda [18], Tomita [19] and Civin-Yood [7] by representation Theory and Bonsall-Duncan [4] by using the Vidav-Palmer characterization of B*-algebras have proved that the second dual of a B*-algebra with the Arens multiplication is a B*-algebra. Bonsall-Duncan have proved even more. They have shown the involution in the second dual is the natural one derived from the involution of the given B*-algebra. We show that if A'' is a B*-algebra under Arens multiplication, then $*$ is invariant on A, and therefore A is a B*-algebra. First we need some definitions and notations.

Let A be a complex unital Banach algebra. Define:

$$D(1) = \{f : f \in A', \|f\| = f(1) = 1\},$$

$$V(A, a) = \{f(a) : f \in A', \|f\| = f(1) = 1\} \quad (a \in A).$$

We say that $h \in A$ is Hermitian if $V(A, R) \subset \mathbb{R}$. We denote the set of all Hermitian elements of A by $H(A)$. A is called a V-algebra if $A = H(A) + iH(A)$. By Proposition 12.20 [6] an element a of a unital B*-algebra is Hermitian if and only if $a^* = a$. Therefore by Lemma 12.3 [6] every unital B*-algebra is a V-algebra. We also denote:

$$H(A') = \{\alpha f - \beta g : f, g \in A';\ \alpha, \beta \in \mathbb{R}^+;\ f(1) = g(1) = \|f\| = \|g\| = 1\}$$
$$= \{\alpha f - \beta g : f, g \in D(1);\ \alpha, \beta \in \mathbb{R}^+\}.$$

<u>1.18 Theorem.</u> Let A be a complex Banach algebra with unit and A'' a B*-algebra under one of the Arens products. Then $*$ is invariant on A.

Proof. Since $\hat{A}$ the natural embedding of A into A'' is a subalgebra of A'', it is enough to prove that $\hat{A}$ is a star subalgebra of A''.

If $\hat{A}$ is not a star subalgebra of A'', then by Lemma 31.9 [5], there exists a $\phi \in A'''$ such that:

$$\phi(\hat{A}) = (0) \quad \text{and} \quad \phi^*(\hat{A}) \neq \{0\} ,$$

where $\qquad \phi^*(F) = [\phi(F^*)]^* \quad (F \in A'')$.

Now A'' is a B*-algebra with unit. Therefore $\|\hat{1}\| = 1$ and so $\|1\| = 1$. i.e. A'' is a unital B*-algebra. But for every B*-algebra B,

$$H(B') = \{f : f(h) \in \mathbb{R}, (h^* = h)\} .$$

Therefore $H((A'')') \cap i\,H((A'')') = \{0\}$. If not, then $\phi' = i\phi''$, and $\phi'(F) \in \mathbb{R} \cap i\mathbb{R} = (0)$, $\phi' = 0$. Also, by Corollary 31.4 [5] we have:

$$A''' = H((A'')') + iH((A'')') .$$

Therefore $\phi = \phi_1 + i\phi_2$, where ϕ_1 and ϕ_2 are in $H((A'')')$. By Lemma 2.6.4 [9], $\phi_1 = \psi_1 - \psi_2$ for some positive linear functionals ψ_1 and ψ_2. Since A'' has unit, by Lemma 37.6 [6]:

$$\psi_K = \psi_K^* \qquad K = 1, 2 .$$

Therefore:

$$\begin{aligned} \phi_1(F) &= \psi_1(F) - \psi_2(F) = \psi_1^*(F) - \psi_2^*(F) \\ &= \phi_1^*(F) = (\phi_1(F^*))^* \qquad F \in A'' . \end{aligned}$$

So:

$$\left\{ \begin{aligned} \phi_1(F^*) &= \phi_1(F)^* \\ \phi_2(F^*) &= \phi_2(F)^* \end{aligned} \right. \qquad F \in A'' . \qquad \text{(I)}$$

But $\phi_1 \in H((A'')')$ gives:

$$\phi_1 = \alpha_1\psi_1' - \alpha_2\psi_2' \qquad \alpha_1, \alpha_2 \in \mathbb{R}^+ ; \quad \psi_1', \psi_2' \in D(\hat{1}) .$$

Clearly: $$\psi_K'\Big|_{\hat{A}} \in D(\hat{A}, \hat{1}) \qquad K = 1, 2$$

and so:

$$\phi_1\Big|_{\hat{A}} \in H((\hat{A})')$$

similarly:

$$\phi_2\Big|_{\hat{A}} \in H((\hat{A})') .$$

Now since $H((A'')') \cap iH((A'')') = \{0\}$, by Hahn-Banach Theorem

$$H((\hat{A})') \cap iH((\hat{A})') = \{0\}$$

Therefore:

$$\phi_1\Big|_{\hat{A}} + i\, \phi_2\Big|_{\hat{A}} = 0 ,$$

$$\phi_1\Big|_{\hat{A}} = \phi_2\Big|_{\hat{A}} = 0 .$$

By (I), $$\phi_K((\hat{A})^*) = (\phi_K(\hat{A}))^* = (0) , \qquad K = 1, 2 .$$

$$\phi_K^*(\hat{A}) = \phi_K((\hat{A})^*)^* = (0) , \qquad K = 1, 2 .$$

$$\phi^*(\hat{A}) = (0)$$

contradiction. Δ

Remarks. 1. Let A be a complex Banach algebra without unit element such that A'' is a B^*-algebra. Then by Lemma 12.19 [6], $A'' + \mathbb{C}$ is a unital Banach algebra. By above Theorem $*$ is invariant on $A + \mathbb{C}$. Again by Lemma 12.19 [6], we get $*$ is invariant on A.

<u>1.19 Corollary</u>. If A" is a B*-algebra, then A is Arens regular.

<u>Proof</u>. By Theorem 18 A is a B*-algebra and by Theorem 7.1 [7], A is Arens regular. Δ

Let A be a Banach algebra and A" a B*-algebra. By Theorem 18, A is a B*-algebra, and by Theorem 1.17.2 [16], A" is a W*-algebra. Therefore, if the second dual of Banach algebra A is a B*algebra, then A" is W*-algebra.

CHAPTER 2

This chapter presents relations between the weakly compact and compact linear operators on a Banach algebra A, associated with the two Arens products defined on A''. Throughout the chapter, the symbols X and Y will denote Banach spaces.

2.1 Definition. Let $T \in BL(X,Y)$, and S be the closed unit sphere in X. The operator T is said to be weakly compact if the weak closure of TS is compact in the weak topology of Y.

2.2 Definition. Let $T \in BL(X,Y)$, and S be the closed unit sphere in X. The operator T is said to be compact if the strong closure of TS is compact in the strong topology of Y.

For $a \in A$, we denote by λ_a and ρ_a the left and right regular representations on A defined by:

$$\lambda_a b = ab \qquad b \in A,$$

$$\rho_a b = ba \qquad b \in A.$$

Consider $\lambda_a^* : A' \to A'$, the adjoint of λ_a. Since for every $f \in A'$ and $b \in A$ we have:

$$\lambda_a^* f(b) = f(\lambda_a b) = f(ab) = fa(b),$$

we get:

$$\lambda_a^* f = fa \qquad (f \in A')$$

Similarly, for $\rho_a^* : A' \to A'$, the adjoint of ρ_a we have:

$$\rho_a^* f = af \qquad (f \in A')$$

Consider $\lambda_a^{**} : A'' \to A''$, the second adjoint of λ_a . Since for every $F \in A''$ and $f \in A'$, we have:

$$\lambda_a^{**}F(f) = F(\lambda_a^{*}f) = F(fa) = F(f\hat{a}) = \hat{a}\#F(f) = \hat{a}F(f) ,$$

we get:

$$\lambda_a^{**}F = \hat{a}F = \hat{a}\#F \qquad (F \in A'')$$

Similarly, for $\rho_a^{**} : A'' \to A''$, the second adjoint of ρ_a we have:

$$\rho_a^{**}F = F\hat{a} = F\#\hat{a} \qquad (F \in A'') .$$

Some parts of Theorem 3 and Corollary 5 have been proved in [20] and [21]. For these parts the proof given here is shorter.

2.3 Theorem. The following statements are equivalent.

(i) $\hat{A}$ is a left (right) ideal in A'' .

(ii) For each $a \in A$, ρ_a (λ_a) is a weakly compact operator on A .

(iii) For each $a \in A$, the mapping $f \to af$ $(f \to fa)$ is a weakly compact operator on A' .

(iv) For each $a \in A$, the mapping $F \to F\hat{a}$ $(F \to \hat{a}F)$ is a weakly compact operator on A'' .

Proof. By Theorem VI.4.2 [10], an operator T in $BL(X,Y)$ is weakly compact if and only if $T^{**}X'' \subset \hat{Y}$. Therefore, for every $a \in A$, ρ_a is weakly compact if and only if $\rho_a^{**}A'' = A''\hat{a} \subset \hat{A}$. Thus $\hat{A}$ is a left ideal of A'' if and only if, for each $a \in A$, ρ_a is weakly compact operator on A . Similar argument can be applied to the right ideal case . Since the operators in (iii) are the adjoint of operators in (ii), and the operators in (iv) are the adjoint of operators in (iii), Gantmacher's Theorem VI.4.8 [10], gives (ii) $\Longleftrightarrow$ (iii) $\Longleftrightarrow$ (iv). Δ

By Theorem 3.1 [22], the natural embedding of every semi-simple annihilator algebra A, is a two-sided ideal of A''. Now, let X be a reflexive Banach space without approximation property, and let $A = KL(X)$ be the algebra of all compact operators on X. Since it contains all bounded operators of finite rank, A obviously operates irreducibly on X, and is therefore semi-simple. By Theorem 2.3 [1], for every $a \in A$, λ_a and ρ_a are weakly compact operators. Therefore $\hat{A}$ is a two-sided ideal in A'. But $A = KL(X)$ is not an annihilator algebra since $FL(X)$ the algebra of finite rank operators on X is a closed two-sided ideal of A, $FL(X) \neq A$ and

$$\operatorname{ran}(FL(X)) = \operatorname{lan}(FL(X)) = (0) .$$

i.e. there exists a semi-simple Banach algebra A such that for every $a \in A$, λ_a and ρ_a are weakly compact, but A is not an annihilator algebra.

<u>2.4 Definition.</u> A subalgebra J of A is called a block subalgebra if:

$$JAJ \subset J .$$

<u>2.5 Corollary.</u> The following statements are equivalent:

(i) $\hat{A}$ is a block subalgebra of A''.

(ii) $\lambda_a \circ \rho_b$ is a weakly compact operator on A for each a and b in A.

(iii) The mapping $f \to bfa$ is a weakly compact operator on A for each a, b in A.

(iv) The mapping $F \to \hat{a}F\hat{b}$ is a weakly compact operator on A'' for each a, b in A.

Proof. For every a and b in A, we have:

$$\lambda_a \circ \rho_b(c) = \lambda_a(\rho_b c) = \lambda_a(cb) = acb \qquad c \in A$$

$$(\lambda_a \circ \rho_b)^*(f) = bfa \qquad f \in A'$$

$$(\lambda_a \circ \rho_b)^{**}(F) = \hat{a}F\hat{b} \qquad F \in A'' .$$

A similar argument to that of Theorem 3 gives the proof of the corollary. Δ

Since $\hat{a}F = \hat{a}\#F$ and $F\hat{a} = F\#\hat{a}$, for every $a \in A$ and $F \in A''$, Theorem 3 and Corollary 5 are also valid, when multiplication in second dual of A is taken to be the second Arens product.

2.6 Proposition Let $a \in A$ and let λ_a be a compact linear operator on A. If $F = w^*\text{-}\lim_\lambda \hat{x}_\lambda$ in A'', for some bounded net $\{x_\lambda\} \subset A$. Then $\|\hat{a}\hat{x}_\lambda - \hat{a}F\| \to 0$.

[redacted]

Proof. By Schauder's Theorem VI.5.2 [10], λ_a is compact if and only if λ_a^* is compact on A'. Now, by Theorem VI.5.6 [10], λ_a^* is compact on A' if and only if its adjoint λ_a^{**} sends bounded nets which converge in the A' topology of A'', into nets which converge in the metric topology of A''. Let $F \in A''$, and $F = w^*\text{-}\lim_\lambda \hat{x}_\lambda$. Then, for every $f \in A'$.

$$\lim_\lambda \hat{x}_\lambda(f) = F(f) .$$

Therefore:

$$\|\lambda_a^{**}\hat{x}_\lambda - \lambda_a^{**}F\| = \|\hat{a}\hat{x}_\lambda - \hat{a}F\| \to 0 . \qquad \Delta$$

Remarks. 1- By similar argument we have: if ρ_a is a compact linear operator on A, then $\|\hat{x}_\lambda \hat{a} - F\hat{a}\| \to 0$ whenever $F \in A''$ and $F = w^*\text{-}\lim_\lambda \hat{x}_\lambda$, $\{x_\lambda\}$ is bounded.

2- By Schauder's Theorem, λ_a is compact on A if and only if λ_a^* is compact on A', and again λ_a^* is compact on A' if and only if λ_a^{**} is compact on A''. Therefore compactness of each of λ_a, λ_a^* and λ_a^{**} on A, A' and A'' respectively, gives: $\|\hat{a}\hat{x}_\lambda - \hat{a}F\| \to 0$, when $\{x_\lambda\}$ is bounded and $F = w^*\text{-}\lim_\lambda \hat{x}_\lambda$.
Similarly compactness of each of ρ_a, ρ_a^* and ρ_a^{**} on A, A' and A'' respectively, gives: $\|\hat{x}_\lambda \hat{a} - F\hat{a}\| \to 0$, whenever $\{x_\lambda\}$ is bounded and $F = w^*\text{-}\lim_\lambda \hat{x}_\lambda$.

<u>2.7 Definition.</u> A minimal idempotent is a non-zero idempotent $e \in A$ such that eAe is a division algebra.

<u>2.8 Example.</u> The following two statements are not equivalent in general.

(i) For every $a \in A$, λ_a is a compact linear operator on A.

(ii) For every $a \in A$, ρ_a is a compact linear operator on A.

<u>Proof.</u> Let B be a Banach algebra which contains minimal idempotents and let e be a minimal idempotent in B such that $\dim Be = \dim A = \infty$. . Then by 31.1 [6], Be is a subalgebra of B.
Now fix $a \in B$. Then by proposition 31.3 [6], there exists $f \in B'$ such that:

$$\lambda_{ae} be = (ae)(be) = a(ebe) = a(e\,be\,e) = a(f(be)e) = f(be)ae \qquad (b \in B)$$

Therefore $\lambda_{ae} = ae \otimes f$, which is of rank ≤ 1 and therefore compact. Now, in case (ii), again fix $a \in B$. Then for some $f \in B'$,

$$\rho_{ae}\, be = be(ae) = b(eae) = b(e\ ae\ e)$$

$$= b(f(ae)e) = be\ f(ae) \qquad b \in B .$$

Therefore $\rho_{ae} = f(ae)I$. Since $KL(Be)$ the set of all compact operators on Be contains $FBL(Be)$, the set of all finite rank operators on Be , then for each $ae \in Be$, λ_{ae} is a compact linear operator on $Be = A$, but in case (ii), they are not. Δ

Again by using Schauder's Theorem, each of the statements in case I as follows is not equivalent in general to any of the statements in case II.

I : For each $a \in A$, $b \to ab$ is compact operator on A .
For each $a \in A$, $f \to fa$ is compact operator on A' .
For each $a \in A$, $F \to \hat{a}F$ is compact operator on A'' .

II : For each $a \in A$, $b \to ba$ is compact operator on A .
For each $a \in A$, $f \to af$ is compact operator on A' .
For each $a \in A$, $F \to F\hat{a}$ is compact operator on A'' .

<u>2.9 Example.</u> Let S be a countable set with the product of two elements defined to be the second element of the pair. Then obviously S is a non-commutative semi-group, and for every $s \in S$, $sS = S$, $Ss = \{s\}$. With convolution as multiplication, consider the Banach algebra $\ell^1(S)$. If $a = \Sigma\alpha_n s_n$, $b = \Sigma\beta_m t_m$ are in $\ell^1(S)$, we have

$$a*b = (\Sigma\alpha_n s_n)(\Sigma\beta_m t_m) = \Sigma\Sigma\alpha_n\beta_m s_n t_m = \Sigma\Sigma\alpha_n\beta_m t_m \quad .$$

Now, let ρ_b be the right regular representation on $\ell^1(S)$. Then:

$$\rho_b a = a*b = \Sigma\alpha_n \ \Sigma\beta_m t_m = \Sigma\alpha_n b \ .$$

Therefore $\rho_b = b \otimes \phi$, where $\phi(a) = \Sigma\alpha_n$, and so ρ_b is a rank one operator on $\ell^1(S)$ and therefore a compact operator. But, for λ_b the left regular representation on $\ell^1(S)$ we have:

$$\lambda_b a = \Sigma\beta_m \ \Sigma\alpha_n s_n = \phi(b)a \ .$$

Therefore $\lambda_b = \phi(b)I$ which is not a compact operator when $\phi(b) \neq 0$. Now, by IV.13.3 [10], in $\ell^1(S)$, weak compact operators and compact operators are the same. Therefore ρ_b is a weakly compact operator, but λ_b is not a weakly compact operator.

If we define the product of S to be the first element of the pair, then λ_b in this case is a compact and therefore a weakly compact operator on $\ell^1(S)$ and ρ_b is not a compact and weakly compact operator on $\ell^1(S)$.

Note that each $s \in S$ is a minimal idempotent of $\ell^1(S)$. Therefore Example 8 would give the "weakly compact" analogue as long as Ae, when $A = \ell^1(S)$, is not reflexive, and we do not need, at this stage the fact that weak compact operators and compact operators on $\ell^1(S)$ are the same.

Let $f \in A'$, denote $\pi_f : A \to A'$ defined by:

$$\pi_f a = fa \ , \qquad a \in A$$

and $\psi_f : A \to A'$ defined by:

$$\psi_f a = af \qquad a \in A \ .$$

Consider $\pi_f^* : A'' \longrightarrow A'$, the adjoint of π_f. Since for every $F \in A''$ and $a \in A$ we have:

$$\pi_f^* F(a) = F(\pi_f a) = F(fa) = Ff(a) ,$$

we get:

$$\pi_f^* F = Ff \qquad F \in A'' .$$

Similarly for $\psi_f^* : A'' \longrightarrow A'$, the adjoint of ψ_f we have:

$$\psi_f^* F = fF \qquad F \in A'' .$$

In the next theorem, (i) $\Longleftrightarrow$ (ii) has been proved for the commutative case by S.L. Gulick, Theorem 3.4 [11], and for the non-commutative case by J. Hennefeld Theorem 2.1 [12]. The proof given here is simpler.

<u>2.10 Theorem.</u> The following are equivalent.

(i) A is Arens regular.

(ii) The mapping $\pi_f : a \to fa$ is a weakly compact operator on A for each $f \in A'$. (Each $f \in A'$ is a weakly almost periodic functional).

(iii) The mapping $\psi_f : a \to af$ is a weakly compact operator on A for each $f \in A'$.

(iv) The mapping $F \to Ff$ is a weakly compact operator on A'' for each $f \in A'$.

(v) The mapping $F \to fF$ is a weakly compact operator on A'' for each $f \in A'$.

<u>Proof</u>. (i) $\Rightarrow$ (ii). Let $f \in A'$. By VI.4.2 [10] it is enough to prove $\pi_f^{**} A'' \subset (A')^\wedge$. Let $F \in A''$. Then for every $G \in A''$;

$$\pi_f^{**} F(G) = F(\pi_f^* G) = F(Gf) = FG(f) = F\#G(f) = (fF)^\wedge(G) .$$

Thus

$$\pi_f^{**}F \in (A')^{\wedge} .$$

(ii) $\Rightarrow$(i). By VI.4.7 [10], $T \in BL(X,Y)$ is weakly compact if and only if $T^* : Y' \to X'$ is continuous with respect to the X'', Y topologies in X', Y' respectively. Take $\{F_\alpha\} \subset A''$ such that $F_\alpha(f) \to F(f)$, $f \in A'$. Then for every $G \in A''$ we have:

$$G(\pi_f^* F_\alpha) \to G(\pi_f^* F) .$$

~~Now let F, G ∈ A" and consider a bounded net~~ $\cancel{\{\gamma_\beta\} \subset A}$ ~~such that~~:

$$\cancel{\lim_\beta \hat{\gamma}_\beta(f) = F(f) \qquad f \in A'.}$$

~~Since~~ $\cancel{\pi_f}$ ~~is weakly compact,~~

$$\cancel{\lim_\beta G(\pi_f^* \hat{\gamma}_\beta) = G(\pi_f^* F) ,}$$

$$G(F_\alpha f) \to G(Ff) ,$$

$$G F_\alpha(f) \to G F(f) .$$

i.e. GF is weak*-continuous in F for fixed G . Now using Theorem 3. 3 [3], we get A is Arens regular.

(i) $\Longleftrightarrow$(iii). By the similar argument mentioned above and using Theorem 3. 3 [3], we get (i)$\Longleftrightarrow$(iii).

To prove (ii)$\Longleftrightarrow$(iv) and (iii)$\Longleftrightarrow$(v) we see that the mappings in (iv) and (v) are the adjoint of the mappings in (ii) and (iii) respectively, and Gantmacher's Theorem VI.4.8 [10] gives the desired conclusion. Δ

Remark. Consider $\lambda_a^* : A' \to A'$, the adjoint of the left regular representation on A , given by $\lambda_a^* f = fa$ and the mapping $\pi_f : A \to A'$ defined by $\pi_f a = fa$. Now, let G be a compact

Hausdorff infinite group. Then by Proposition 4.1 [20] $[L^1(G)]^\wedge$ is a two-sided ideal in $[L^1(G)]''$, thus λ_a^* is a weakly compact operator on $[L^1(G)]'$ for each $a \in L^1(G)$. But, by [24], since G is infinite, $L^1(G)$ is not Arens regular. Thus π_f is not a weakly compact operator on $L^1(G)$ for every $f \in [L^1(G)]'$. Also consider Example 9. By Theorem 2 [23], $\ell^1(S)$, when $st = t$ or $st = s$ $(s,t \in S)$ is Arens regular. But λ_a^* is not a weakly compact operator on A' for every $a \in A$, when the product of S is defined by $st = t$ $(s, t \in S)$, and ρ_a^* is not a weakly compact operator on A' for every $a \in A$, when the product of S is defined by $st = s$ $(s, t \in S)$, i.e. there is no relation in general between the operators π_f on A $(f \in A')$, and λ_a^* on A' $(a \in A)$ as far as weak compactness is concerned.

<u>2.11 Corollary</u>. Let A be commutative. Then A is Arens regular if and only if:

$$\pi_f^{**}F = \psi_f^{**}F = (Ff)^\wedge \qquad f \in A',\ F \in A''.$$

<u>Proof</u>. Let A be Arens regular. Then for every $G \in A''$ we have

$$\pi_f^{**}F(G) = F(\pi_f^{*}G) = FG(f) = F\#G(f) = G(fF)$$

$$= (fF)^\wedge(G) = (Ff)^\wedge(G) \qquad f \in A' \quad F \in A''$$

Thus:

$$\pi_f^{**}F = (Ff)^\wedge .$$

Conversely, let $\pi_f^{**}F = (Ff)^\wedge$ for every $f \in A'$ and $F \in A''$. Then

$$FG(f) = F(Gf)\ \text{[redacted]} = (Gf)^\wedge(F)$$

$$= \pi_f^{**}G(F) = G(\pi_f^{*}F) = G(fF) = F\#G(f) .$$

for every F, G in A'' and $f \in A'$. Thus A is Arens regular.

Note that for commutative algebra A , $\pi_f = \psi_f$. Δ

Remark. Let f be a multiplicative linear functional on A . Then by argument of lemma 3.6 [7] :

$$fF = Ff = F(f)f \qquad F \in A''$$

Therefore, for every $F, G \in A''$ we have:

$$F\#G(f) = G(fF) = G(F(f)f) = G(f)F(f)$$

$$FG(f) = F(Gf) = F(G(f)f) = F(f)G(f)$$

i.e. the two Arens products coincide on Φ_A the set of multiplicative linear functionals on A . Note that, if f is a multiplicative linear functional, then π_f and ψ_f are compact operators and therefore they are weakly compact operators, and by argument of Theorem 10, again we get that the two Arens products coincide on Φ_A .

2.12 Definition. A linear functional $f \in A'$ is said to be an almost periodic functional if $\{fa : ||a|| \le 1\}^-$ is compact in A' .

The next Theorem essentially has been proved by S.A. McKilligan and A.J. White 2.2 [14], but the argument given here is shorter.

2.13 Theorem. The following are equivalent:

(i) For every $f \in A'$, π_f is a compact linear operator on A .
(Every $f \in A'$ is almost periodic functional.)

(ii) For every $f \in A'$, ψ_f is a compact linear operator on A .

(iii) For every $f \in A'$, $F \to Ff$ is a compact linear operator on A'' .

(iv) For every $f \in A'$, $F \to fF$ is a compact linear operator on A'' .

(v) For every $f \in A'$, if $F_\alpha(g) \to F(g)$ $(g \in A')$ where $\{F_\alpha\} \subset A''$ is bounded, then $||F_\alpha f - Ff|| \to 0$.

(vi) For every $f \in A'$, if $F_\alpha(g) \to F(g)$ $(g \in A')$, where $\{F_\alpha\} \subset A''$ is bounded, then $||fF_\alpha - fF|| \to 0$.

(vii) For every F, G in A'', $(F, G) \to FG$ is jointly bounded weak*-continuous.

(viii) For every F, G in A'', $(F, G) \to F\#G$ is jointly bounded weak*-continuous.

Proof. Since the maps in (iii) are the adjoints of the maps in (i), by Schauder's Theorem V.5.2 [10], (i) $\Leftrightarrow$ (iii).

Similarly (ii) $\Longleftrightarrow$ (iv).

(i) $\Longleftrightarrow$ (v). Let $f \in A'$. Then by VI.5.6 [10], π_f is compact if and only if its adjoint π_f^*, sends bounded nets which converge in the A' topology of A'' into nets which converge in the metric topology of A'. Thus π_f is compact if and only if $||\pi_f^* F_\alpha - \pi_f^* F|| = ||F_\alpha f - Ff|| \to 0$, whenever $\{F_\alpha\}$ is a bounded net in A'', and $F_\alpha(g) \to F(g)$ $\forall g \in A'$.

Similarly (ii) $\Longleftrightarrow$ (vi).

(v) $\Longrightarrow$ (vii). Let $F, G \in A''$, $F_\alpha(g) \to F(g)$ $(\forall g \in A')$, $G_\beta(g) \to G(g)$ $(\forall g \in A')$, where $\{F_\alpha\}$ and $\{G_\beta\}$ are bounded nets in A'' and let $f \in A'$. Then

$$|F_\alpha G_\beta(f) - FG(f)| = |F_\alpha G_\beta(f) - F_\alpha G(f) + F_\alpha G(f) - FG(f)|$$
$$\leq |F_\alpha(G_\beta f) - F_\alpha(Gf)| + |F_\alpha(Gf) - F(Gf)|$$
$$\leq ||F_\alpha|| \, ||G_\beta f - Gf|| + |F_\alpha(Gf) - F(Gf)|$$

Now, $\{F_\alpha\}$ is bounded, $||G_\beta f - Gf|| \to 0$ by hypothesis, and $|F_\alpha(Gf) - F(Gf)| \to 0$.

(vii) $\Rightarrow$ (v). Let $\{F_\lambda\} \subset A''$ be a bounded net in A'' and $F_\lambda(g) \to F(g)$ $\forall g \in A'$. We have to prove:

$$\lim_\lambda \| F_\lambda f - Ff \| \to 0$$

i.e. $\sup_{\|a\| \leq 1} |F_\lambda f(a) - Ff(a)| \to 0$

i.e. $\sup_{\|a\| \leq 1} |\hat{a}F_\lambda(f) - \hat{a}F(f)| \to 0$.

Suppose otherwise and let

$$\sup_{\|a\| \leq 1} |\hat{a}F_\lambda(f) - \hat{a}F(f)| \not\to 0 .$$

Then there exists $\varepsilon > 0$ and a subnet $\{F_{\lambda_K}\}$ such that:

$$\sup_{\|a\| \leq 1} |\hat{a}F_{\lambda_K}(f) - \hat{a}F(f)| \geq \varepsilon$$

Therefore we can find $\{a_{\lambda_K}\} \subset A$ such that $\|a_{\lambda_K}\| \leq 1$ and

$$|\hat{a}_{\lambda_K} F_{\lambda_K}(f) - \hat{a}_{\lambda_K} F(f)| \geq \varepsilon/2 .$$

But the closed unit ball of A'' is weak*-compact. Let G be a weak*-cluster point of $\{\hat{a}_{\lambda_K}\}$. Since multiplication in A'' is jointly bounded weak*-continuous, $\{\hat{a}_{\lambda_K} F_{\lambda_K}\}$ has GF as weak*-cluster point. Thus

$$|\hat{a}_{\lambda_K} F_{\lambda_K}(f) - \hat{a}_{\lambda_K} F(f)|$$

can be made as small as we please. Contradiction.

Similarly we can prove (vi) $\Longleftrightarrow$ (vii).

To complete the proof we have to prove (vii) $\Longleftrightarrow$ (viii). Let (vii) hold, and let $F, G \in A''$. Then by (v) for every $f \in A'$:

$$\|\hat{y}_\lambda f - Gf\| \to 0$$

when $G = w^*\text{-}\lim_\lambda \hat{y}_\lambda$, and $\{y_\lambda\} \subset A$ is bounded. Therefore:

$$FG(f) = F(Gf) = \lim_\lambda F(\hat{y}_\lambda f) = \lim_\lambda F\hat{y}_\lambda(f)$$

$$= \lim_\lambda F\#\hat{y}_\lambda(f) = \lim_\lambda \hat{y}_\lambda(fF) = G(fF) = F\#G(f)$$

i.e. A is Arens regular. Therefore for every F, G in A''

$$(F, G) \to FG = F\#G$$

is jointly bounded weak*-continuous.

(viii) $\Longrightarrow$ (vii). Let $F, G \in A''$. Again we have (viii) $\Longleftrightarrow$ (vi). Therefore for every $f \in A'$ we have:

$$\|f\hat{x}_\alpha - fF\| \to 0$$

when $F = w^*\text{-}\lim_\alpha \hat{x}_\alpha$ and $\{x_\alpha\} \subset A$ is bounded. Therefore:

$$F\#G(f) = G(fF) = \lim_\alpha G(f\hat{x}_\alpha) = \lim_\alpha G(fx_\alpha)$$

$$= \lim_\alpha Gf(x_\alpha) = \lim_\alpha \hat{x}_\alpha(Gf) = \lim_\alpha \hat{x}_\alpha G(f) = FG(f) .$$

i.e. A is Arens regular. Therefore for every F, G in A''

$$(F, G) \to F\#G = FG$$

is jointly bounded weak*-continuous. Consequently all implications are proved. Δ

<u>Remark.</u> 1. A is Arens regular, if one of the conditions in Theorem 13 is valid. Actually from (i), if π_f is a compact linear operator on A for every $f \in A'$, then it is weakly compact and Theorem 10 gives A is Arens regular.

2. Again we can prove that, there is no relation in general between the operators ρ_a^* on A' defined by $\rho_a^* f = af$ and ψ_f

on A defined by $\psi_f a = af$ as far as compactness is concerned. For, let A be a Banach algebra which contains a minimal idempotent e with $\dim Ae = \infty$. Similar to Example 8, Ae is a subalgebra of A, $\pi_f = f \otimes \mathcal{X}'$ and $\psi_f = \mathcal{X}' \otimes f$ for some $\mathcal{X}' \in A'$. i.e. for all $f \in A'$, π_f and ψ_f are rank one operators and so compact. Now by Example 8, for every $a \in A$, λ_a and therefore λ_a^* is compact, but ρ_a and therefore ρ_a^* is not compact.

For $F \in A''$, let $T_F : A' \to A'$ be defined by:

$$T_F f = Ff, \qquad f \in A',$$

and $S_F : A' \to A'$ defined by

$$S_F f = fF \qquad f \in A'.$$

Consider $T_F^* : A'' \to A''$, the adjoint of T_F. Since for every $G \in A''$ and $f \in A'$ we have:

$$T_F^* G(f) = G(T_F f) = G(Ff) = GF(f),$$

we get:

$$T_F^* G = GF \qquad G \in A''.$$

Similarly, for $S_F^* : A'' \to A''$, the adjoint of S_F we have:

$$S_F^* G = F\#G \qquad G \in A''.$$

2.14 Theorem. The following are equivalent:

(i) For every $F \in A''$, T_F is a weakly compact operator on A'.

(ii) The mapping $G \to GF$ is a weakly compact operator on A'' for every $F \in A''$.

(iii) $(A'')^\wedge$ is a left ideal in $A^{(4)}$, the fourth dual of A, when A'' has the first Arens product.

<u>Proof</u>. Since the maps in (ii) are the adjoint of those in (i), Gantmacher's Theorem VI.4.8 [10], gives (i) $\Longleftrightarrow$ (ii).

To prove (i) $\Longleftrightarrow$ (iii), $T_F^{**} : A''' \to A'''$ is defined by:

$$T_F^{**}\ \phi(G) = \phi(T_F^{*}G) = \phi(GF) = F\phi(G) \qquad \phi \in A''' ,\ G \in A'' .$$

Therefore:

$$T_F^{**}\phi = F\phi \qquad \phi \in A'''$$

And $$T_F^{***}W = W\hat{F} \qquad W \in A^{(4)} .$$

Now, Theorem VI.4.2 [10], gives that: T_F^{*} is weakly compact on A'' for every $F \in A''$, if and only if: $A^{(4)}\hat{F} \subset (A'')^{\wedge}$. Λ

Note that, for Banach algebra B if $b \in B$ and $F \in B''$, then $\hat{b}F = \hat{b}\#F$. Therefore $(A'')^{\wedge}$ is a left ideal of $A^{(4)}$ (with respect to each of the two Arens products in $A^{(4)}$ arisen from the first Arens produce in A''), if and only if T_F is a weakly compact operator on A' for each $F \in A''$.

Similarly we can prove that the following are equivalent:

(i) For every $F \in A''$, S_F is a weakly compact operator on A' .

(ii) The mapping $G \to F\#G$ is a weakly compact operator on A'' for every $F \in A''$.

(iii) $(A'')^{\wedge}$ is a right ideal in $A^{(4)}$, when A'' has the second Arens product.

Again $(A'')^{\wedge}$ is a right ideal of $A^{(4)}$ with respect to each of the two Arens products in $A^{(4)}$ arisen from the second Arens product in A'' .

By Theorem 14, T_F^{*} is a weakly compact operator for every $F \in A''$ if and only if [redacted] $(A'')^{\wedge}$ is a left ideal in $A^{(4)}$. Therefore by Theorem 3 we get T_F for every $F \in A''$ is weakly compact if and only if ρ_F, the right regular representation on A'', is weakly compact. (A'' with the first Arens product.) Similarly S_F is weakly compact for every $F \in A''$ if and only if λ_F the left regular representation on A'' is weakly compact. (A'' with the second Arens product.) Moreover we have:

<u>2.15 Corollary.</u> Let A be Commutative. Then λ_F and ρ_F, the left and the right regular representations on A'', with respect to the first Arens product are weakly compact if and only if they are weakly compact with respect to the second Arens product.

The condition: For every $F \in A''$, T_F is a weakly compact operator on A' in Theorem 14 is indeed a very strong condition. Next we give an example for which T_F and S_F for every $F \in A''$ are compact and therefore weak compact on A', and A is Arens regular.

<u>2.16 Example.</u> Let $A = \ell^1$, the space of absolutely convergent series of complex numbers, with its usual norm, and let multiplication in A be defined co-ordinatewise. Then by Theorem 4.2 [22], Theorem 4.2 [7] and Theorem 3.10 [7],

$$A'' = \hat{A} \oplus \mathrm{rad}(A'') = \hat{A} \oplus M^{\perp} = \hat{A} \oplus \mathrm{ran}(A'') ,$$

where M is the closed subspace of ℓ^{∞} generated by multiplicative linear functionals on A. Since A is commutative and Arens regular, then A'' is commutative and therefore:

$$T_F = S_F \qquad F \in A''$$

Let $B = \mathrm{ran}(A'')$. Then

$$A^{(4)} = A'' \oplus B'' = A \oplus B \oplus B'' .$$

By this construction and considering that $B = \mathrm{ran}(A'') = \mathrm{ran}(\hat{A})$, we get $(A'')^{\wedge}$ is an ideal of $A^{(4)}$. i.e. For every $F \in A''$, T_F and S_F are weakly compact operators on A'. Now by IV.13.3 [10] compact and weak compact operators on A are the same.

2.17 Corollary. If T_F is a weakly compact operator on A' for every $F \in A''$, then $\hat{A}$ is a left ideal of A''.

Proof. Let ρ_a be the right regular representation on A. Then

$$\rho_a^{**} G = G\hat{a} \qquad G \in A''$$

But $T_{\hat{a}}^{*} G = G\hat{a}$; $G \in A''$, $a \in A$.

Therefore: $\qquad T_{\hat{a}}^{*} = \rho_a^{**} \qquad a \in A$.

Theorem 3 gives the result. Δ

Similarly we have: If S_F is a weakly compact operator on A' for every $F \in A''$, then $\hat{A}$ is a right ideal of A''.

Remark. Let G be a compact abelian group. Then by Theorem 4.1 [20], $\hat{A} = [L^1(G)]^{\wedge}$ is a two sided ideal of $A'' = [L^1(G)]''$. We prove that A'' is not an ideal of $A^{(4)}$. Suppose otherwise and let R be the radical of A''. We prove that $A''/_R$ is an ideal of $(A''/_R)''$. But $(A''/_R)'' \approx A^{(4)}/_{R^{\perp\perp}}$. Since for every $F \in A''$ and $\phi \in A^{(4)}$ we have:

$$(\hat{F} + R^{\perp\perp})(\phi + R^{\perp\perp}) = \hat{F}\phi + \hat{F}R^{\perp\perp} + R^{\perp\perp}\phi + R^{\perp\perp}$$

But $\hat{F}R^{\perp\perp} \subset R^{\perp\perp}$. And for each $G \in R^{\perp\perp}$, if $P \in R^{\perp}$ we have $P(G) = 0$. Thus:

$$\hat{G}\phi(P) = \phi(PG) = \lim_{\lambda} \hat{\eta}_{\lambda}(PG) = \lim_{\lambda} P(G\eta_{\lambda})$$

where $\phi = w^{*}\text{-}\lim_{\lambda} \hat{\eta}_{\lambda}$ for bounded net $\{\eta_{\lambda}\} \subset A''$. But $G\eta_{\lambda} \in R$, $P(G\eta_{\lambda}) = 0$, $G\phi \in R^{\perp\perp}$. i.e. $(\hat{F} + R^{\perp\perp})(\phi + R^{\perp\perp}) = \hat{F}\phi + R^{\perp\perp}$, $(A''/_R)'' \approx A^{(4)}/_{R^{\perp\perp}}$. By Theorem 3.17 [7] $A''/_R$ is isometrically isomorphic to the measure algebra of G, and by Theorem 5 [21], $[M(G)]^{\wedge}$ is a two-sided ideal of $[M(G)]''$ if and only if G is finite. i.e. there exists a Banach algebra A such that λ_a and ρ_a are weakly compact for every $a \in A$, but there exists $F \in A''$ such that T_F and S_F are not weakly compact on A'.

<u>2.18 Proposition.</u> Let $F, G \in A''$, T_F be a compact operator on A'' and $G = w^*\text{-}\lim_{\beta} \hat{y}_{\beta}$, when $\{y_{\beta}\} \subset A$ is bounded. Then $\|\hat{y}_{\beta}F - GF\| \to 0$ ~~(or GF is continuous from the weak* topology to norm topology).~~

<u>Proof.</u> By Theorem VI.5.6 [10], T_F is compact on A' if and only if its adjoint T_F^* sends bounded nets which converge in the A' topology of A'' into nets which converge in the metric topology of A''. Now:

$$G(f) = \lim_{\beta} \hat{y}_{\beta}(f) \qquad f \in A'$$

Therefore:

$$\|T_F^*\hat{y}_{\beta} - T_F^*G\| = \|\hat{y}_{\beta}F - GF\| \to 0 . \qquad \Delta$$

Similarly we can get, if S_F is compact on A' and $G = w^*\text{-}\lim_{\beta} \hat{y}_{\beta}$. Then:

$$\|F\hat{y}_{\beta} - F\#G\| \to 0 .$$

<u>2.19 Proposition.</u> If T_F is compact on A', then ρ_F the right regular representation on A'', when A'' has the first Arens product is compact. If S_F is compact on A', then λ_F the left regular representation on A'', when A'' has the second Arens product, is compact.

CHAPTER 3

In this chapter the second dual of Banach annihilator algebras are studied.

Let E be a subset of a complex Banach algebra A. The left and right annihilators of E are the sets $\mathrm{lan}(E)$, $\mathrm{ran}(E)$ given by:

$$\mathrm{lan}(E) = \{x \in A : xE = (0)\}$$

$$\mathrm{ran}(E) = \{x \in A : Ex = (0)\} .$$

__3.1 Definition.__ A Banach algebra A is said to be an annihilator algebra if it satisfies the following axioms:

For all closed left ideals L and closed right ideals R:

(i) $\mathrm{ran}(L) = 0$ if and only if $L = A$,

(ii) $\mathrm{lan}(R) = 0$ if and only if $R = A$.

__3.2 Definition.__ A Banach algebra A is a dual algebra if for each closed left ideal L and each closed right ideal R:

$$\mathrm{lan}(\mathrm{ran}(L))) = L , \quad \mathrm{ran}(\mathrm{lan}(R)) = R .$$

It is obvious that every dual algebra is an annihilator algebra.

__3.3 Proposition.__ Let A be a semi-simple annihilator algebra. Then every minimal left (right) ideal of $\hat{A}$ is a minimal left (right) ideal of A''.

__Proof.__ Let $L(R)$ be a minimal left (right) ideal of A. By Proposition 30.6 [6], $L = Ae$ $(R = eA)$ where e is a minimal idempotent of A. Now, since A is semi-simple annihilator algebra, by Theorem 3.1 [22], $\hat{A}$ is a closed two sided ideal of

A'' . Therefore:

$$A''\hat{e} \subset \hat{A} \qquad (\hat{e}A'' \subset \hat{A}) ,$$

$$A''\hat{e} = A''\hat{e}\hat{e} \subset \hat{A}\hat{e} \qquad (\hat{e}\hat{A}'' = \hat{e}\hat{e}A'' \subset \hat{e}\hat{A}) ,$$

$$\hat{e}\hat{A}''\hat{e} \subset \hat{e}\hat{A}\hat{e} = \mathbb{C}\,\hat{e} \qquad (\hat{e}A''\hat{e} \subset \hat{e}\hat{A}\hat{e} = \mathbb{C}\,\hat{e}) .$$

Thus: $\hat{e}A''\hat{e} = \hat{e}\hat{A}\hat{e}$, i.e. $\hat{e}$ is a minimal idempotent of A'' , for which $\hat{L} = A''\hat{e}$ $(\hat{R} = \hat{e}A'')$.

Now applying Proposition 30.6 [6], we get $\hat{L}(\hat{R})$ is a minimal left (right) ideal if A'' . Δ

<u>3.4 Proposition.</u> Let A be a semi-simple annihilator algebra. Then with respect to the first Arens product the following are equivalent:

(i) $\operatorname{ran}_{A''}(\hat{A}) = (0)$

(ii) $\operatorname{ran}(A'') = (0)$

(iii) A'' is semi-simple.

<u>Proof.</u> To prove (i)$\Longleftrightarrow$(ii), it is enough to show that, for every Banach algebra A, $\operatorname{ran}_{A''}(\hat{A}) = \operatorname{ran}(A'')$. Since $\hat{A} \subset A''$, $\operatorname{ran}(A'') \subset \operatorname{ran}_{A''}(A)$. Let $G \in \operatorname{ran}_{A''}(A)$, $F \in A''$ and $F = w^*\text{-}\lim_{\alpha} \hat{x}_{\alpha}$, where $\{x_{\alpha}\}$ is a bounded net in A . Since $(F, G) \to FG$ is weak*-continuous in F for fixed G , we have:

$$FG = w^*\text{-}\lim_{\alpha} \hat{x}_{\alpha} G = 0$$

Then $\operatorname{ran}_{A''}(\hat{A}) = \operatorname{ran}(A'')$.

(ii)$\Longleftrightarrow$(iii). By Theorem 4.1 [22], we have:

$$\operatorname{rad}(A'') = \{F \in A'' : A''F = (0)\} = \operatorname{ran}(A'') . \quad \Delta$$

Note that in proposition 4 the product for A'' was the first Arens product. By similar argument when A'' has the second Arens product, the following are equivalent:

$$\operatorname{lan}_{A''}(\hat{A}) = (0) \ ;$$

$$\operatorname{lan}(A'') = (0) \ ;$$

$$A'' \text{ is semi-simple.}$$

By Theorem 4.1 [22], for a semi-simple annihilator algebra A, the two radicals of A'' coincide. Thus, A'' with respect to each of the Arens product is semi-simple if and only if one of the following holds:

$$\operatorname{ran}(A'') = \operatorname{ran}_{A''}(\hat{A}) = (0) \qquad \text{in first product.}$$

$$\operatorname{lan}(A'') = \operatorname{lan}_{A''}(\hat{A}) = (0) \qquad \text{in second product.}$$

<u>3.5 Theorem.</u> Let A be a semi-simple annihilator algebra. Then A'' is an annihilator algebra if and only if A is reflexive.

<u>Proof.</u> Let A'' be an annihilator algebra. Then by Proposition 4, A'' is semi-simple, and since by Theorem 3.1 [22], $\hat{A}$ is a two-sided ideal in A'', by Lemma 32.4 [6], we get:

$$A'' = (\hat{A} \oplus \operatorname{ran}_{A''}(\hat{A}))^{-} \ .$$

Considering Proposition 4, we get $A'' = \hat{A}$. The converse is obvious. Δ

Note that, in Theorem 5 to get $A'' = (\hat{A} \oplus \operatorname{ran}_{A''}(\hat{A}))^{-}$, we need to have: A'' is semi-prime annihilator algebra. By an elementary argument, without using Proposition 4, and therefore Theorem 4.1 [22], we can get this as follows:

<u>3.6 Lemma.</u> Let A be a semi-simple annihilator algebra. If $\operatorname{ran}(A'') = (0)$, then A'' is semi-prime.

<u>Proof.</u> Let J be a two-sided ideal in A'' such that $J^2 = (0)$. Let L be a minimal left ideal of A''. Then $J \cap L = (0)$, or

$J \cap L = L$. In both cases we have $JL = (0)$. Thus

$$\mathrm{soc}(A'') \subset \mathrm{ran}(J) ,$$

$$\mathrm{soc}(\hat{A}) \subset \mathrm{ran}(J) .$$

By Corollary 32.6 [6], $A = (\mathrm{soc}(A))^-$. Therefore:

$$J \subset \mathrm{ran}_{A''}(\hat{A}) = \mathrm{ran}(A'') = (0) ,$$

$$J = (0) . \quad \Delta$$

3.7 Theorem. Let A'' be a semi-simple annihilator algebra. Then $A'' = A$.

Proof. By Theorem 3.1 [22], A'' is a two-sided ideal in $A^{(4)}$. Therefore by Theorem 2.3, λ_F and ρ_F , the left and right regular representations on A'', are weakly compact for every $F \in A''$. Thus $\lambda_{\hat{a}}$ and $\rho_{\hat{a}}$ are weakly compact operators on A'' for every $a \in A$. But $\lambda_{\hat{a}} = {\lambda'_a}^{**}$ and $\rho_{\hat{a}} = {\rho'_a}^{**}$, where λ'_a and ρ'_a are the left and right regular representations on A , for each $a \in A$. Therefore $\hat{A}$ is a two-sided ideal in A'' . But by Lemma 32.4 [6],

$$A'' = (\hat{A} \oplus \mathrm{ran}_{A''}(\hat{A}))^- ,$$

and since: $\mathrm{ran}_{A''}(\hat{A}) = \mathrm{ran}(A'') = (0)$

we get $A'' = \hat{A}$. Δ

Since every W*-algebra has identity element, and since every semi-simple annihilator algebra is finite dimensional if and only if it has identity element, we get that, every annihilator W*-algebra is finite dimensional.

3.8 Corollary. Let A'' be an annihilator B*-algebra. Then A'' is finite dimensional.

Proof. By Theorem 1.18, A is a B*-algebra. Therefore A" is a W*-algebra and is an annihilator algebra. Δ

Next we give an example of a topologically simple reflexive annihilator star algebra which has an unbounded approximate identity, and it can not have any one sided bounded approximate identity.

3.9 Example. Let H be a separable Hilbert space, and $\{u_n\}$, $\{v_m\}$ be any pair of complete orthonormal systems of vectors in H. By Parseval's equality, it is easy to show that for every $T \in BL(H)$:

$$\sum_n |Tu_n|^2 = \sum_{n,m} |(Tu_n, v_m)|^2 = \sum_m |T^* v_m|^2 .$$

This common value will be denoted by $|T|\phi^2$. The Schmidt-class $F\phi$ consists of all those operators $T \in BL(H)$, such that

$$|T|\phi < \infty .$$

By A 1.3 [15], $F\phi$ is a topologically simple reflexive annihilator Banach star algebra which can be identified with an infinite matrix algebra M_Λ. Now, since H is separable, the cardinal number of index set Λ is $\aleph_0$. We shall prove that the sequence of all infinite matrices e_n defined by:

$$e_n = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & 0 & \\ & & & \ddots & & & \\ & & & & 1 & & \\ & 0 & & & & 0 & \\ & & & & & & 0 \\ & & & & & & & \ddots \end{pmatrix}$$

is a two-sided approximate identity (not bounded) for this infinite matrix algebra. Let

$$a = \{a_{n,m}\}_{n,m=1}^{\infty} = \left(\begin{array}{c|c} A_n & B \\ \hline C & D \end{array}\right)$$

such that

$$\|a\| = \left[\sum_{n,m=1}^{\infty} |a_{nm}|^2\right]^{\frac{1}{2}} < \infty$$

Consider

$$e_n = \left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & 0 \end{array}\right) .$$

Then

$$a - ae_n = \left(\begin{array}{c|c} 0 & B \\ \hline 0 & D \end{array}\right) ,$$

and

$$a - e_n a = \left(\begin{array}{c|c} 0 & 0 \\ \hline C & D \end{array}\right) .$$

Therefore: $\|a - ae_n\| \to 0$ as $n \to \infty$, and $\|a - e_n a\| \to 0$ as $n \to \infty$. Thus $F\phi$ has a two-sided approximate identity (not bounded) with above properties.

Now, by Corollary 28.8 [6], for Arens regular Banach algebra A, A'' has unit, if and only if A has bounded two-sided approximate identity. Thus every reflexive Banach algebra with bounded two-sided approximate identity has unit. Now, let $\{T_\lambda\}$ be a bounded left approximate identity. Since $F\phi$ is reflexive, it has left identity E. But, for every $T \in F\phi$:

$$0 = \sum_n |(ET - T)u_n|^2 = \sum_{n,m} |((ET - T)u_n, v_m)|^2$$

$$= \sum_{n,m} |(u_n, (T^*E^* - T^*)v_m)|^2 .$$

Therefore E is a right identity. Similarly, if E is a right identity in $F\phi$, then it is a left identity. But annihilator algebras with identity are finite dimension.

Note that, Example 9 can be modified for non-separable Hilbert space H .

3.10 Example. By Proposition 34.4 [6], any semi-simple H^*-algebra is an annihilator algebra, and we thus get a class of reflexive annihilator Banach algebras which have approximate identity, but are not finite dimensional.

3.11 Proposition. Let A be a semi-simple commutative annihilator algebra, let M_A be its carrier space, and M the closed subspace of A' spanned by M_A . Let:

$$M^{\perp} = \{F \in A''; F(M) = (0)\} .$$

Then $\mathrm{rad}(A'') = (A'/_{M})'$ and M is the closed linear subspace of A' spanned by $Q = \{fa : f \in A', a \in A\}$.

Proof. By Corollary 4.2 [22], $\mathrm{rad}\, A'' = M^{\perp}$, and by II.4.18b [10], $M^{\perp}$ and $(A'/_{M})'$ are isometrically isomorphic. Therefore $\mathrm{rad}(A'') = (A'/_{M})'$. Now, $\mathrm{rad}(A'') = \mathrm{ran}(A'') = M^{\perp}$, and by Theorem 3.10 [7], we have:

$$A''M^{\perp} = (0) \Longleftrightarrow \{fa : a \in A,\ f \in A'\} \subset M .$$

So $Q \subset M$. Let f be a multiplicative linear functional on A . Then for every a, b in A :

$$f(ab) = f(a)f(b) = fa(b) .$$

Let $a \in A$, with $f(a) \neq 0$. Then:

$$fa = f(a)f ,$$

$$f = \frac{1}{f(a)}\, fa .$$

i.e. f is in Q , and completes the proof. Λ

By Theorem 4.2 [7], $A = \ell^1$, the space of absolutely convergent series of complex numbers, with its usual norm, and multiplication defined co-ordinatewise is a commutative semi-simple annihilator algebra, such that A'' is commutative but not semi-simple and

$$A'' = \hat{A} \oplus \mathrm{rad}(A'') . \qquad \mathrm{I}$$

Next we will prove this for the non commutative case and we give a commutative semi-simple annihilator algebra A such that A'' does not satisfy I.

Let $\{B_n\}_{n=1}^{\infty}$ be a sequence of semi-simple annihilator algebras, such that, for every $n \in \mathbb{N}$, $B'_n B_n$ is dense in B'_n. Consider $A = \ell^1(B_n)$, with pointwise addition, scalar multiplication and product. Define the norm $\|\ \|$ on A by:

$$\|a\| = \|\{a_n\}\| = \sum_{n=1}^{\infty} \|a_n\|_n ,$$

where $a_n \in B_n$, $\|\ \|_n$ is the norm in B_n $(n = 1, 2, \ldots)$.

<u>3.12 Lemma.</u> $A = \ell^1(B_n)$ is a semi-simple annihilator algebra.

<u>Proof.</u> Consider the projection $\pi_i : \ell^1(B_n) \to B_i$ defined by:

$$\pi_i(\{a_n\}) = a_i \qquad (i = 1, 2, \ldots) .$$

Since each B_i is semi-simple, $\pi_i \circ \sigma_i$ is an irreducible representation on A for an irreducible representation σ_i on B_i. Now:

$$\ker(\pi_i \circ \sigma_i) = \{a = \{a_n\} \in A : a_i = 0\} .$$

And:

$$\mathrm{rad}(A) \subset \bigcap_{i=1}^{\infty} \ker(\pi_i \circ \sigma_i) = (0) .$$

Therefore A is semi-simple.

To prove that A is annihilator algebra, consider:

$$U_i = \{0, 0, \ldots, 0, B_i, 0, \ldots\} .$$

Then U_i is a closed two-sided ideal of A. Since A is the topological sum of the semi-simple annihilator algebras U_i $(i = 1, 2, \ldots)$, by Theorem 2.8.29 [15], A is an annihilator algebra. Δ

3.13 Lemma. The closed linear span of $A'A$ is $C_0(B'_n)$.

Proof. Consider $A' = \ell^\infty(B'_n)$. Let $f = \{f_n\} \in A'$, $a = \{a_n\} \in A$, where, $f(a) = \sum_n f_n(a_n)$.

For every $x = \{x_n\} \in A$, we have:

$$fa(x) = f(ax) = \sum_n f_n(a_n x_n) = \sum_n f_n a_n(x_n) .$$

Therefore: $$fa = \left\{f_n a_n\right\} .$$

But:

$$\|fa\| = \sup_n \|f_n a_n\| ,$$

and since B'_n is a Banach B_n-module:

$$\|f_n a_n\|_n \leq \beta \|f_n\|_n \|a_n\|_n \leq \beta \|f\| \|a_n\|_n$$

for a positive β. But $\|a_n\|_n \to 0$ as $n \to \infty$. Therefore $fa \in C_0(B'_n)$. Now, fix $b_n \in B_n$, $f_n \in B'_n$. Then:

$$(0, 0, \ldots, 0, f_n b_n, 0, \ldots) \in A'A .$$

Since for every $n \in \mathbb{N}$, $B'_n B_n$ is dense in B'_n, the closed linear span of $A'A = \{fa : f \in A', a \in A\}$ is $C_0(B'_n)$. Δ

3.14 Lemma. $\mathrm{rad}(A'') = (A'A)^\perp$.

Proof. By Lemma 4.1 [22], for every semi-simple annihilator algebra A:

$$\mathrm{rad}(A'') = \mathrm{ran}(A'') .$$

By Proposition 4: $\mathrm{ran}(A'') = \mathrm{ran}_{A''}(\hat{A})$.

Therefore:

$$\operatorname{rad}(A'') = \operatorname{ran}_{A''}(\hat{A}) = \{F \in A'' : \hat{A}F = (0)\}$$

$$= \{F \in A'' : \hat{A}F(A') = (0)\} = \{F \in A'' : F(A'A) = (0)\}$$

$$= (A'A)^{\perp} . \quad \Delta$$

3.15 Lemma. Let B_n for every $n \in \mathbb{N}$ be a dual algebra. Then $A = \ell^1(B_n)$ is a dual algebra.

Proof. Let $a = \{b_n\} \in A = \ell^1(B_n)$. Since each B_n is a dual algebra, 2.8.3 [15], $b_n \in \overline{b_n B_n}$, $(n \in \mathbb{N})$. So for every $n \in \mathbb{N}$, there exists a sequence $\{\beta_{nm}\} \subset B_n$ such that:

$$b_n \beta_{nm} \to b_n \quad \text{as} \quad m \to \infty .$$

Let $\alpha_{km} = (\beta_{1m}, \beta_{2m}, \ldots, \beta_{km}, 0, \ldots)$. Then obviously $\alpha_{km} \in A$,

$$a\alpha_{km} = (b_1\beta_{1m}, b_2\beta_{2m}, \ldots, b_k\beta_{km}, 0, \ldots) ,$$

$$a\alpha_{km} \longrightarrow (b_1, b_2, \ldots, b_k, 0, \ldots) .$$

So, $(b_1, b_2, \ldots, b_k, 0, \ldots) \in \overline{aA}$ for each $k \in \mathbb{N}$. Similarly, $(b_1, b_2, \ldots, b_k, 0, \ldots) \in \overline{Aa}$ for each $k \in \mathbb{N}$. Now, consider:

$$a_1 = \{b_1, 0, 0, \ldots\}$$

$$a_2 = \{b_1, b_2, 0, 0, \ldots\}$$

$$\cdots\cdots\cdots\cdots$$

$$a_k = \{b_1, b_2, \ldots, b_k, 0\} .$$

Then, since $a_k \in \overline{aA}$ and $a_k \to a$, we have $a \in \overline{aA}$. Similarly $a \in \overline{Aa}$. Therefore $a \in \overline{aA} \cap \overline{Aa}$. Now, Theorem 2.8.29 [15] gives that A is a dual algebra.

3.16 Theorem. Let $A = \ell^1(B_n)$, where $\{B_n\}$ is a sequence of semi-simple annihilator algebras, such that $B'_n B_n$ is dense in B'_n for every $n \in \mathbb{N}$. Then:

$$A'' = \hat{A} \oplus \mathrm{rad}(A'') = \hat{A} \oplus [C_0(B'_n)]^{\perp} = \hat{A} \oplus P^{\perp}.$$

where P is the closed linear span of $A'A$.

Proof. By Lemma 12, A is a semi-simple annihilator algebra, and by Lemma 13, P the closed linear span of $A'A$ can be identified with $C_0(B'_n)$, considered as a subspace of $\ell^{\infty}(B'_n)$. The topology $\sigma(A, P)$ is then the same topology on A as its w*-topology, where A is considered as the dual of $C_0(B_n)$. Since P is total, and since Alaoglu's Theorem asserts that the unit ball of A is compact in $\sigma(A, P)$, Theorem 4.1 [7] gives:

$$A'' = \hat{A} \oplus P^{\perp}.$$

And by Lemma 13 and Lemma 14, we have:

$$A'' = \hat{A} \oplus \mathrm{rad}(A'') = \hat{A} \oplus C_0(B'_n)^{\perp}. \quad \Delta$$

Case 1. Let $A = \ell^1(M_{k_n}(\mathbb{C}))$, $(k_n \in \mathbb{N})$, with pointwise addition, scalar multiplication and product. Define the norm $\|\ \|$ on A by:

$$\|a\| = \|\{a_{k_n}\}\| = \sum_{n=1}^{\infty} |a_{k_n}|,$$

where $a_{k_n} \in M_{k_n}(\mathbb{C})$, $|\ |$ is operator norm in $M_{k_n}(\mathbb{C})$, and $\mathbb{C}^{k_n}$ has ℓ^1-norm. Note that, in this case $M_{k_n}(\mathbb{C})$ is a dual algebra for each $n \in \mathbb{N}$. Thus by Lemma 15, $A = \ell^1(M_{k_n}(\mathbb{C}))$ is a dual algebra. Moreover, the module multiplication fa defined by: $fa(b) = f(ab)$ $(b \in A)$, can be characterized with the multiplication of matrices. Indeed we have: Let

$$a = \left\{ \left[a_{ij}^{k_n} \right] \right\}_{n=1}^{\infty} \in A, \quad f = \left\{ \left[f_{ij}^{k_n} \right] \right\}_{n=1}^{\infty} \in A'.$$

Then:

$$fa = \left\{ \left[a_{ji}^{k_n} \right] \left[f_{ij}^{k_n} \right] \right\}_{n=1}^{\infty}$$

since for every

$$b = \left\{ \left[b_{ij}^{k_n} \right] \right\}_{n=1}^{\infty} \in A$$

we have:

$$ab = \left\{ \left[\Sigma_{m=1}^{k_n} a_{im}^{k_n} b_{mj} \right] \right\}_{n=1}^{\infty} \quad .$$

Then:

$$f(ab) = \sum_{n=1}^{\infty} \sum_{i=1}^{k_n} \sum_{j=1}^{k_n} \sum_{m=1}^{k_n} (a_{im}^{k_n} b_{mj}^{k_n} f_{ij}^{k_n}) = fa(b) \ .$$

Now, let $d = \left[d_{rs}^{k_n} \right]$ be the transpose of $\left[a_{rs}^{k_n} \right]$. $n = 1, 2, \ldots$.

Then:

$$d\times f = \left\{ \left[\sum_{t=1}^{k_n} d_{rt}^{k_n} f_{ts} \right] \right\}_{n=1}^{\infty} \quad .$$

Now, for every $b = \left\{ \left[b_{rs}^{k_n} \right] \right\}_{n=1}^{\infty} \in A$, consider,

$$\sum_{n=1}^{\infty} \Sigma_{r=1}^{k_n} \Sigma_{s=1}^{k_n} \Sigma_{t=1}^{k_n} (a_{tr}^{k_n} f_{ts}^{k_n} b_{rs}^{k_n}) \ ,$$

and take $t = i$, $r = m$, and $s = j$, we get that $d\times f$ has acted on A as fa . i.e. $A'A$ consists of elements:

$$fa = d\times f \qquad f \in A', \ a \in A, \ d \in A \ ,$$

where the multiplication in left hand side is the module multiplication defined on A by: $fa(b) = f(ab)$, and the multiplication in right hand side is the pointwise multiplication of the two sequences of matrices, and the terms of d are the transposes of the terms of a . Therefore $A'A = A'$.

<u>Case 2</u>. Let $A = \ell^1(M_{k_n}(\mathbb{C}), w_n)$, with pointwise addition, scalar multiplication and product, where $\{w_n\}$ is a sequence of positive real numbers with:

$$w_{m+n} \le w_m w_n \qquad m, n \in \mathbb{N} .$$

Define the norm $\|\ \|$ on A by:

$$\|a\| = \|\{a_{k_n} w_n\}\| = \sum_{n=1}^{\infty} \|a_{k_n} w_n\|_n ,$$

where $a_{k_n} \in M_{k_n}(\mathbb{C})$, $\|\ \|_n$ is operator norm in $M_{k_n}(\mathbb{C})$, and $\mathbb{C}^{k_n}$ has ℓ^1-norm. Again we have: A is a dual algebra with $A'' = \hat{A} \oplus \mathrm{rad}(A'')$.

<u>Case 3</u>. Let B_n be the Schmidt-class $F\phi_n$ of operators on separable Hilbert space $H_n (n = 1, 2, \ldots)$, Example 9 . Then by A 1.3 [15] each B_n can be identified with an infinite matrix algebra of order $\aleph_0$. To prove each $B'_n B_n$ is dense in B'_n , let:

$$f = \left(\begin{array}{c|c} C_m & D \\ \hline G & H \end{array}\right) \in B'_n .$$

Then:

$$\left\| f - \left(\begin{array}{c|c} C_m & 0 \\ \hline 0 & 0 \end{array}\right) \right\| \to 0 \quad \text{as} \quad m \to \infty .$$

Now, since for every finite matrix algebra B , $B'B$ is dense in B' , we get $B'_n B_n$ is dense in B'_n . Also the same argument of case 1 can be applied to characterize the elements of $A'A$.

Note that case 3 can be modified for non-separable Hilbert spaces H_n .

<u>Case 4</u>. Let B_n be a semi-simple H*-algebra. Since $B'_n \approx B_n$, by Theorem 4.10.31 [15], B'_n is equal to the topological direct sum of its minimal-closed-two-sided ideals $\Sigma^{\oplus}_{\Lambda_n} I'_\lambda$, where each I'_λ is a topologically simple H*-algebra. But by Theorem 4.10.32 [15], each topologically simple H*-algebra I'_λ is bicontinuously isomorphic with an infinite matrix algebra M'_λ. By case 3, $M'_\lambda M_\lambda$ is dense in M'_λ. So to prove $B'_n B_n$ is dense in B'_n we need to prove that, for every

$$f = \{f_\lambda\}_{\lambda\in\Lambda_n} \in B'_n \ , \quad a = \{a_\lambda\}_{\lambda\in\Lambda_n} \in B_n \ ,$$

$$fa = \{f_\lambda a_\lambda\}_{\lambda\in\Lambda_n} \ .$$

Let:

$$b = \{b_\lambda\}_{\lambda\in\Lambda_n} \in B_n \ .$$

Then:

$$fa(b) = f(ab) = f(\{a_\lambda b_\lambda\}) = \Sigma\ f_\lambda(a_\lambda b_\lambda)$$

$$= \Sigma\ f_\lambda a_\lambda(b_\lambda) = \{f_\lambda a_\lambda\}(b) \ .$$

Therefore:

$$fa = \{f_\lambda a_\lambda\}_{\lambda\in\Lambda_n} \ .$$

<u>Case 5</u>. Let $B_n = KL(H_n)$, the algebra of compact operators on Hilbert space H_n. Then $B'_n = TC(H_n)$, the trace class of operators on H_n. By A.1.4 [15], $FKL(H_n)$, the algebra of finite rank operators on H_n is dense in $TC(H_n)$. Therefore to prove $B'_n B_n$ is dense in B'_n, since $FKL(H_n)$ is dense in $B_n = KL(H_n)$, it is enough to show that, every $f \in FKL(H_n)$ can be written as $f = fp$, when p is finite rank projection in $KL(H)$.

Let $f = u \otimes \nu$ be of rank one. Then to prove $f = fp$, since

$$f(a) = \mathcal{T}(au \otimes \nu) = (au, \nu), \qquad a \in B_n$$

we have:

$$((pa - a)u, \nu) = 0 \qquad a \in B_n$$

$$\Leftrightarrow \quad (pau, \nu) = (au, \nu) \qquad a \in B_n$$

$$\Leftrightarrow \quad (au, p\nu) = (au, \nu) \qquad a \in B_n$$

Now, take $p = \frac{1}{\|\nu\|^2} \nu \otimes \nu$, then $p\nu = \nu$. Therefore $B'_n B_n$ is dense in B'_n. Δ

<u>3.17 Corollary.</u> Under the conditions of Theorem 16, $A = \ell^1(B_n)$ is Arens regular.

<u>Proof.</u> Let $F, G \in A''$. Then $F = \hat{a} + F_1$, $G = \hat{b} + G_1$ when $a, b \in A$ and $F_1, G_1 \in \mathrm{rad}(A'')$. But:

$$FG = \hat{a}\hat{b} + \hat{a}G_1 + F_1\hat{b} + F_1G_1 = \hat{a}\hat{b} + F_1\hat{b} = \hat{a}\hat{b} + F_1\#\hat{b}\ .$$

But, by Theorem 4.1 [22]:

$$\mathrm{rad}(A'') = \{F \in A'' : A''F = (0)\} = \{F \in A'' : F\#A'' = (0)\}\ .$$

Therefore: $F_1 \# \hat{b} = 0$, $FG = \hat{a}\hat{b}$. Similarly $F\#G = \hat{a}\hat{b}$. Δ

<u>3.18 Corollary.</u> There exists a commutative semi-simple dual algebra A such that: $A'' \neq \hat{A} \oplus \mathrm{rad}(A'')$.

<u>Proof.</u> Let G be an abelian compact group. Then $A = L^1(G)$ is a semi-simple commutative dual algebra. Now if $A'' = \hat{A} \oplus \mathrm{rad}(A'')$, then by above corollary A is Arens regular. But by [24], A is Arens regular if and only if G is finite. Δ

By considering Theorem 4.1 [22], since for semi-simple annihilator algebra A, R_1^{**}, the radical of A'' with respect to the first Arens product coincides with R_2^{**} the radical of A'' with respect to second Arens product, if $A = \ell^1(B_n)$, when B_n is semi-simple annihilator algebra and $B_nB'_n$ is dense in B'_n $(n = 1, 2, \ldots)$, then A is a semi-simple annihilator algebra and $A'' = \hat{A} \oplus \text{rad}(A'')$.

3.19 Definition. A compact Banach algebra is a compact algebra A, such that for each $t \in A$, the mapping $a \to tat$ is a compact linear operator on A.

It follows from Lemma 33.12 [6], that every semi-simple annihilator algebra is a compact Banach algebra. By Theorem 5, the second dual of a semi-simple annihilator algebra is an annihilator algebra if and only if A is reflexive. This case can not occur for compact Banach algebras. Indeed we have:

3.20 Theorem. There exists a non-reflexive semi-simple compact commutative Banach algebra A, such that A'' is compact and not semi-simple.

Proof. Let $A = \ell^1$, the algebra of absolutely convergent series of complex numbers, with usual norm, and let multiplication in A be defined co-ordinatewise. By Lemma 33.12 [6], A is compact, and by Theorem 4.2 [7], $A'' = \hat{A} \oplus \text{ran}(A'')$. Now, let $G \in A''$. Then $G = \hat{a} + \phi$, where $a \in \ell^1$ and $\phi \in \text{ran}(A'')$. Define the mapping $\rho_G : A'' \to A''$ by:

$$\rho_G F = GFG \qquad F \in A''.$$

Then:

$$\rho_G F = GFG = \hat{a}F\hat{a} + \hat{a}F\phi + \phi F\phi + \phi F\hat{a}.$$

Since $\phi \in \text{ran}(A'')$, $\hat{a}F\phi = \phi F\phi = 0$. Now by Theorem 3.1 [22],

$\hat{A}$ is a two-sided ideal in A''. Therefore $F\hat{a} = \hat{b}$ for some $b \in A$, and since A is commutative:

$$\phi F\hat{a} = \phi\hat{b} = \hat{b}\phi = F\hat{a}\phi = \hat{a}F\phi = 0 .$$

Therefore:

$$\rho_G F = \hat{a}F\hat{a} \qquad (G = \hat{a} + \phi) , \quad F \in A'' .$$

Now, define $\rho'_a : A \to A$ by:

$$\rho'_a b = aba \qquad b \in A .$$

Then:

$$\rho'^*_a f = afa \qquad (f \in A') .$$

$$\rho'^{**}_a F = \hat{a}F\hat{a} \qquad (F \in A'') .$$

Since ρ'_a is compact on A, by Schauder's Theorem IV.5.2 [10], ρ'^{**}_a is compact on A'', and therefore $\rho_G = \rho'^{**}_a$ is compact, i.e. A'' is a compact Banach algebra. Δ

Note that, by Theorem 5, the second dual of a semi-simple annihilator algebra A is annihilator algebra if and only if A is reflexive. And every semi-simple annihilator algebra is a compact Banach algebra. But, let $A = KL(H)$. Then A is non-reflexive semi-simple compact Banach algebra while A'' is semi-simple, but not annihilator algebra.

CHAPTER 4

Let S be a semigroup and consider $\ell^1(S)$ the semigroup algebra of S. In this chapter we particularize some of the problems in Chapters 2 and 3 to the Banach algebra $\ell^1(S)$.

4.1 Theorem. Let S be a semigroup. Then the following statements are equivalent:

(i) sS is finite for every $s \in S$.

(ii) λ_a, the left regular representation on $\ell^1(S)$, is a compact operator for every $a \in \ell^1(S)$.

(iii) λ_a, the left regular representation on $\ell^1(S)$, is a weakly compact operator for every $a \in \ell^1(S)$.

Proof. (i) $\Rightarrow$(ii). Let $a \in \ell^1(S)$. Then $a = \sum_{n=1}^{\infty} \alpha_n s_n$ when $\alpha_n = a(s_n)$. Consider λ_{s_n}. Since $s_n S$ is finite $\lambda_{s_n}(\ell^1(S))$ is a finite dimension subspace of $\ell^1(S)$. Therefore λ_{s_n} is a compact operator on $\ell^1(S)$ and we have:

$$\sum_{n=1}^{N} \alpha_n \lambda_{s_n} \qquad N \in \mathbb{N}$$

is compact. But:

$$\lambda_a = \sum_{n=1}^{\infty} \alpha_n \lambda_{s_n} .$$

Now by Lemma VI.5.3 [10], the set of compact operators is closed in the uniform operator topology of $BL(X, Y)$ and we get λ_a is compact operator on $\ell^1(S)$.

(ii) $\Rightarrow$ (i). Let $s_0 \in S$ and $s_0 S$ be infinite. Then there ecists $u_n \in S$ such that $n \to s_0 u_n$ is one-one. Therefore λ_{s_0}

is isometric on an infinite dimension subspace of $\ell^1(S)$, i.e. λ_{s_0} is not compact. Contradiction.

(ii) $\Longleftrightarrow$ (iii). By Corollary IV.8.14 [10], weak and strong convergence of sequences in $\ell^1(S)$ are the same. Thus (ii) $\Longleftrightarrow$ (iii).

Remark. Similarly we can prove that for a semigroup S the following are equivalent:

(i) Ss is finite for every $s \in S$.

(ii) ρ_a is compact on $\ell^1(S)$ for every $a \in \ell^1(S)$.

(iii) ρ_a is weakly compact on $\ell^1(S)$ for every $a \in \ell^1(S)$.

And if Ss and sS are finite for every $s \in S$, then for every a, b in $\ell^1(S)$, $\lambda_a \circ \rho_b$ is a compact operator on $\ell^1(S)$, and therefore $\ell^1(S)$ is a compact Banach algebra.

4.2 Theorem. If $\ell^1(S)$ is semi-simple, then the following are equivalent:

(i) $(sS) \cup (Ss)$ is finite for every $s \in S$ and $S = \{st : s, t \in S\}$.

(ii) $\ell^1(S)$ is an annihilator algebra.

Proof. (i) $\Longrightarrow$ (ii). Let $s \in S$. Since $(tS) \cup (St)$ is finite for each $t \in S$, SsS is finite and therefore $\ell^1(S)\, s\, \ell^1(S)$ is finite dimensional. Since $\ell^1(S)$ is semi-simple and $\ell^1(S)\, s\, \ell^1(S)$ is an ideal of $\ell^1(S)$, $\ell^1(S)\, s\, \ell^1(S)$ is a semi-simple finite dimensional ideal of $\ell^1(S)$. Therefore $\ell^1(S)\, s\, \ell^1(S)$ is isomorphic with the direct sum of full matrix algebras. Now using Theorem 2.8.29 [15], we get $\ell^1(S)\, s\, \ell^1(S)$ is an annihilator algebra. Now, let $p \in S$. Then $p = s_1 s_2$ for some s_1 and s_2 in S , and $s_2 = t_1 t_2$ for some t_1 and t_2 in S . Thus:

$$p = s_1 t_1 t_2 \in St_1 S \quad \text{for some} \quad t_1 \in S \ .$$

Since for every element $a \in \ell^1(S)$ we have:

$$a = \sum_{n=1}^{\infty} \alpha_n p_n \qquad p_n \in S \quad \text{where} \quad a(p_n) = \alpha_n$$

we get that $\ell^1(S)$ is the topological sum of full matrix algebras, and again by 2.8.29 [15], $\ell^1(S)$ is an annihilator algebra.

(ii) $\Rightarrow$(i). Since $\ell^1(S)$ is a semi-simple annihilator algebra, by Theorem 3.1 [22], $[\ell^1(S)]^\wedge$ is a two-sided ideal in its second dual space. So by Theorem 2.3 λ_a and ρ_a are weakly compact on $\ell^1(S)$ for every a, b in $\ell^1(S)$. Theorem 1 gives sS and Ss are finite for every $s \in S$. To prove $S = \{st : s, t \in S\}$, we have

$$\ell^1(S)\ \ell^1(S) \subset \ell^1(S^2)$$

where $\ell^1(S^2)$ is a closed two-sided ideal of $\ell^1(S)$. Now $\ell^1(S)$ is an annihilator algebra, therefore

$$\begin{aligned} & \operatorname{ran}(\ell^1(S)) = (0) \\ \Rightarrow\ & \operatorname{ran}(\ell^1(S))^2 = (0) \\ \Rightarrow\ & \operatorname{ran}(\ell^1(S^2)) = (0) \\ \Rightarrow\ & \ell^1(S^2) = \ell^1(S) \\ \Rightarrow\ & S^2 = S \ . \quad \Delta \end{aligned}$$

4.3 Theorem. (Young) The following are equivalent for any locally compact Hausdorff semi-topological semigroup S .

(i) $\ell^1(S)$ has regular multiplication.

(ii) There is no pair of sequences $\{x_n\}$, $\{y_m\}$ in S such that the sets:

$$\{x_n y_m : n > m\} \quad \text{and} \quad \{x_n y_m : m > n\}$$

are disjoint.

<u>Proof.</u> ([23] Theorem 2).

<u>4.4 Corollary.</u> There exists a countable semigroup S such that for every $s \in S$, sS is finite and $\ell^1(S)$ is commutative but not Arens regular.

<u>Proof.</u> Let $S = \mathbb{N}$, and define

$$mn = \min\{m, n\} \qquad m, n \in \mathbb{N} .$$

Obviously S is a commutative semigroup and $n\mathbb{N}$ and $\mathbb{N}n$ are finite for every $n \in \mathbb{N}$. Now let $\{x_n\}$ be the sequence of positive odd integers and $\{y_m\}$ the sequence of positive even integers. Then

$$\{x_n y_m : n > m\} = \{y_m\}$$

and

$$\{x_n y_m : m > n\} = \{x_n\} .$$

Therefore:

$$\{x_n y_m : m > n\} \cap \{x_n y_m : n > m\} = \phi .$$

Using Theorem 3, we get $\ell^1(S)$ is not Arens regular. Now since S is commutative, $\ell^1(S)$ is commutative. Δ

<u>4.5 Corollary.</u> Let S be a semigroup containing (i) an infinite group or (ii) an infinite chain of idempotents. Then $\ell^1(S)$ is not Arens regular.

<u>Proof.</u> Let G be an infinite subgroup of S. Then $\ell^1(G)$ is a closed subalgebra of $\ell^1(S)$. Now, if $\ell^1(S)$ is Arens regular, then by 6.3 [7], $\ell^1(G)$ is Arens regular, and by [24], we get G is finite.

Let $E_S = \{s_1, s_2, \dots\}$ be an infinite lower chain of idempotents in S. Then $E_S \approx (\mathbb{N}, \wedge)$. Corollary 4 gives $\ell^1(E_S)$

is not Arens regular. Therefore $\ell^1(S)$ is not Arens regular. A similar argument deals with the case of upper chains, in which case we use $(\mathbb{N}, \vee)$. Δ

4.6 Definition. A semigroup S is an inverse semigroup if for any $s \in S$, there exists a unique $s^* \in S$ such that

$$s\, s^*\, s = s \quad \text{and} \quad s^*\, s\, s^* = s^* .$$

4.7 Proposition. There exists an infinite inverse semigroup S with $\ell^1(S)$ Arens regular.

Proof. Let S be an infinite semigroup of idempotents with product defined by

$$st = \theta \qquad s, t \in S, \quad s \neq t$$

Then obviously S is an inverse semigroup, and since for every sequence $\{x_n\}$ and $\{y_m\}$ in S

$$\{x_n y_m : m > n\} \cap \{x_n y_m : n > m\} = \{\theta\} ,$$

Theorem 3 gives $\ell^1(S)$ is Arens regular. Δ

We define the Brandt semigroup S over a group G with index set I to be the semigroup consisting of elementary $I \times I$-matrices over $G \cup \{0\}$ and the zero matrix θ. We write

$$S = \{(g)_{ij} : g \in G \cup \{0\} , \quad i, j \in I\} \cup \{\theta\}$$

and we have:

$$(g)_{ij}(h)_{kl} = \begin{cases} (g\,h)_{il} & \text{if } j = K \\ \theta & \text{if } j \neq K \end{cases}$$

Brandt semigroups are inverse semigroups.

4.8 Theorem. If S is a Brandt semigroup, then $\ell^1(S)$ is not Arens regular.

Proof. Consider the sequences $\{x_n\}$ and $\{y_m\}$ defined by:

$$x_n = (g)_{n1} = \begin{bmatrix} 0 & 0 & \cdots \\ 0 & & \\ \vdots & & \\ e_{n1} & & 0 \\ 0 & & \\ \vdots & & \end{bmatrix}, \quad y_m = (g)_{1m} = \begin{bmatrix} 0 & 0 & \cdots & e_{1m} & 0 & \cdots \\ 0 & & & & & \\ 0 & & & & & \\ \vdots & & & 0 & & \\ \vdots & & & & & \end{bmatrix}$$

with $e_{n1} \neq 0$, $e_{1m} \neq 0$; $m, n \in \mathbb{N}$.

Then:

$$x_n y_m = (g)_{nm}$$

$$\{x_n y_m : n > m\} = \{(g)_{nm} : n > m\}$$

$$\{x_n y_m : m > n\} = \{(g)_{nm} : m > n\}$$

Therefore $\{(g)_{nm} : n > m\} \cap \{(g)_{nm} : m > n\} = \emptyset$.

By using Theorem 3, we get $\ell^1(S)$ is not Arens regular. Δ

Note that if the group of the Brandt semigroup S is trivial then S contains neither an infinite subgroup nor an infinite chain of idempotents.

4.9. Corollary. If S is a semigroup containing a Brandt semigroup then $\ell^1(S)$ is not Arens regular.

<u>Problem 1.</u> Characterize the semigroup S such that each $\phi \in \ell^{\infty}(S)$ is almost periodic.

Let $\phi \in \ell^{\infty}(S)$. If ϕ is almost periodic then it is weakly almost periodic and by Theorem 2.10 we get $\ell^1(S)$ is Arens regular. In particular by Corollary 4, the condition that $Ss \cup sS$ be finite for each $s \in S$ is not sufficient.

<u>Problem 2.</u> Characterize the semigroup S such that each irreducible representation of $\ell^1(S)$ is finite dimensional.

Clearly the commutative case is trivial.

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