Classical Logic Through the Looking-Glass

PETER MILNE

Abstract: In Lewis Carroll's *Through the Looking Glass and What Alice Found There*, Alice enters through a mirror into the realm reflected. It is, of course, left-right reversed but this is only the start of the fun and games when Alice explores the world on the other side of the mirror. Borrowing, if only in part, Carroll's theme of inversion, my aim is to take a look at classical logic in something of an inverted way, or, to be more exact, in three somewhat inverted ways. Firstly, I come at proof of the completeness of classical logic in the Lindenbaum–Henkin style backwards: I take for granted the existence of a set Σ for which it holds, for some formula ϕ, that $\psi \notin \Sigma$ if, and only if, $\Sigma \cup \{\psi\} \vdash \phi$ then read off the rules of inference governing connectives and quantifiers that most directly yield the desired (classical) semantic properties. We thus obtain *general elimination rules* and what I have elsewhere called *general introduction rules*. Secondly, the same approach lets us read off a different set of rules: those of the cut-free sequent calculus $\mathscr{S}'$ of (Smullyan, 1968). Smullyan uses this calculus in proving the Craig–Lyndon interpolation theorem for first-order logic (without identity and function symbols). By attending very carefully to the steps in Smullyan's proof, we obtain a strengthening: if $\phi \vdash \psi$, $\nvdash \neg\phi$ and $\nvdash \psi$ then there is an interpolant χ, a formula employing only the non-logical vocabulary common to ϕ and ψ, such that ϕ entails χ in the first-order version of Kleene's 3-valued logic and χ entails ψ in the first-order version of Graham Priest's Logic of Paradox. The result, which is hidden from view in natural deduction formulations of classical logic, extends, I believe, to first-order logic with identity. Thirdly, we look at a contraction-free "approximation" to classical propositional logic. Adding the general introduction rules for negation or the conditional leads to Contraction being a derived rule, apparently blurring the distinction between structural and operational rules.

Keywords: Completeness proof for classical first-order logic, Lindenbaum–Henkin construction, general elimination rules, general introduction rules, Craig interpolation lemma, Kleene's strong three-valued logic, Logic of Paradox, Łukasiewicz's infinite-valued logic, Contraction

1 Introduction

Speaking of A. J. Ayer's criticism of Sartre's philosophy of *le néant*, the Norwegian philosopher Arne Næss said,

> Characteristically the critic's appeal in this case is not to our scientific sensibilities and the logical calculus of predicates but to that much respected tribunal in British philosophy, *Alice through the Looking Glass*. (Næss, 1968, p. 318)

To give it its proper title, this work, the second of the Reverend Charles Lutwidge Dodgson's Alice books, is *Through the Looking-Glass and What Alice Found There*. Alice enters through a mirror into the realm reflected. It is, of course, left-right reversed but this is only the start of the fun and games when Alice explores the world on the other side of the mirror.

I shall not invoke *Through the Looking-Glass* as a tribunal (however, exactly, one might do that). I want only to borrow, and only in part, Lewis Carroll's theme of inversion.[1] My aim is to take a look at classical logic in something of an inverted, back-to-front way, or, to be more exact, in three, distinct but related, somewhat back-to-front ways.

First I'll come at the Lindenbaum–Henkin proof of completeness backwards, obtaining rules with a view to showing that the maximal, consistent extension has the properties required of the set of formulas. Then I'll look at a refinement of the Craig–Lyndon Interpolation Theorem for Classical First-Order Logic using a formulation of first-order logic derived from the first part. Thirdly, I'll develop what can reasonably be called a contraction-free variant of Classical First-Order Logic. Replacing standard rules for negation or the conditional with the rules obtained in the first investigation, *i.e.*, by changing what would normally be thought of as operational rules, we re-introduce contraction (at the cost of proofs without the subformula property).

2 Ideal rules for proving completeness[2]

In outline the orthodox procedure when proving the completeness of classical logic in the Lindenbaum–Henkin style is as follows:

[1]On the looking-glass theme and inversion in Carroll's writings, see (Carroll, 1970, n. 4, pp. 180–83).

[2]This section is in part a reworking of material drawn from (Milne, 2008, 2010, 2015).

1. Starting from a (possibly empty) set of premises, Σ, and a formula, χ, that is *not* derivable from the set and given an enumeration of formulas in the language, one expands the premise set successively adding just those formulas whose addition does not lead to the derivability of the initially underivable sentence.

2. In the limit this procedure yields a set of sentences Σ_∞ for which, for all formulas ϕ of the language,

$$\phi \notin \Sigma_\infty \text{ if, and only if, } \Sigma_\infty \cup \{\phi\} \vdash \chi.$$

3. We then use the rules of the logic to show that Σ_∞ has exactly the closure properties it would have were it the set of formulas true (satisfied) in a model. As $\psi \notin \Sigma_\infty$, there is, then, a model in which all members of Σ are true and χ is not.

Let us go at this from the opposite end—Carrollian inversion! Suppose that we have a set of formulas possessing the closure properties of a set of formulas true (satisfied) in a model as classically understood. What would be the best rules for showing this?

You may think that this isn't a very precise question so let me show you what I have in mind. Consider conjunction, first of all. Here we need that $\phi \wedge \psi \in \Sigma_\infty$ if, and only if, $\phi \in \Sigma_\infty$ and $\psi \in \Sigma_\infty$. First, I'll contrapose this: $\phi \wedge \psi \notin \Sigma_\infty$ if, and only if, $\phi \notin \Sigma_\infty$ or $\psi \notin \Sigma_\infty$. Next, I'll recast this in terms of the Lindenbaum–Henkin condition of non-membership:

- if $\Sigma_\infty \cup \{\phi\} \vdash \chi$ then $\Sigma_\infty \cup \{\phi \wedge \psi\} \vdash \chi$;
- if $\Sigma_\infty \cup \{\psi\} \vdash \chi$ then $\Sigma_\infty \cup \{\phi \wedge \psi\} \vdash \chi$;
- if $\Sigma_\infty \cup \{\phi \wedge \psi\} \vdash \chi$ then $\Sigma_\infty \cup \{\phi, \psi\} \vdash \chi$.

What we are looking for are generally applicable rules of inference that guarantee this as directly as possible. What we read off the first condition is that when ϕ together with side premises entails χ then $\phi \wedge \psi$ together with those same side premises also entails χ. Likewise, from the second condition we read off that when ψ together with side premises entails χ then $\phi \wedge \psi$ together with those same side premises also entails χ. We set these out as follows:

$$\dfrac{\phi \wedge \psi \qquad \begin{matrix}[\phi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\, m \;\; \wedge\text{-elimination (l)}; \qquad \dfrac{\phi \wedge \psi \qquad \begin{matrix}[\psi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\, m \;\; \wedge\text{-elimination (r)}.$$

These are no more than transcriptions of the standard elimination rules in what is sometimes called the *general elimination* format. (See (Milne, 2015, p. 192) for references.) Terminology: we say that the conjunction occurs *categorically* and that the conjuncts occur *hypothetically* in these rules.

We read off the third condition that when $\phi \wedge \psi$ together with side premises entails χ then those same side premises together with ϕ and ψ suffice to entail χ. We may write this thus:

$$\dfrac{\phi \qquad \psi \qquad \begin{matrix}[\phi \wedge \psi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\, m \;\; \wedge\text{-introduction.}$$

(Here the conjunction occurs hypothetically, the conjuncts categorically.)

From this, which promises to be the most straightforward case, we see that introduction rules are very different in form from the norm. Usually, the introduced connective occurs as main connective in a formula which stands as conclusion of the application of the rule:

$$\dfrac{\phi \qquad \psi}{\phi \wedge \psi} \; .$$

In our rule the introduced connective occurs as main connective in a formula which is an assumption apt for discharge in the application of the rule. In effect, the standard rule is the special case when χ is $\phi \wedge \psi$.

In a small step in the direction of familiarity, disjunction gives us these conditions:

- if $\Sigma_\infty \cup \{\phi\} \vdash \chi$ and $\Sigma_\infty \cup \{\psi\} \vdash \chi$ then $\Sigma_\infty \cup \{\phi \vee \psi\} \vdash \chi$;
- if $\Sigma_\infty \cup \{\phi \vee \psi\} \vdash \chi$ then $\Sigma_\infty \cup \{\phi\} \vdash \chi$;
- if $\Sigma_\infty \cup \{\phi \vee \psi\} \vdash \chi$ then $\Sigma_\infty \cup \{\psi\} \vdash \chi$;

and hence these rules:

$$\dfrac{\phi \qquad \begin{matrix}[\phi \vee \psi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\; m \;\; \vee\text{-introduction (l)}; \qquad \dfrac{\psi \qquad \begin{matrix}[\phi \vee \psi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\; m \;\; \vee\text{-introduction (r)};$$

$$\dfrac{\phi \vee \psi \qquad \begin{matrix}[\phi]^m \\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\psi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\; m \;\; \vee\text{-elimination.}$$

The standard $\vee$-elimination rule is already in general elimination format.

In the case of the conditional, we get something really different. First, we have that $\phi \to \psi \in \Sigma_\infty$ if, and only if, $\phi \notin \Sigma_\infty$ or $\psi \in \Sigma_\infty$. Contraposing, $\phi \to \psi \notin \Sigma_\infty$ if, and only if, $\phi \in \Sigma_\infty$ and $\psi \notin \Sigma_\infty$. Hence, recasting in terms of the condition of non-membership:

- if $\Sigma_\infty \cup \{\psi\} \vdash \chi$ then $\Sigma_\infty \cup \{\phi, \phi \to \psi\} \vdash \chi$;
- if $\Sigma_\infty \cup \{\phi \to \psi\} \vdash \chi$ then $\phi \in \Sigma_\infty$;
- if $\Sigma_\infty \cup \{\phi \to \psi\} \vdash \chi$ then $\Sigma_\infty \cup \{\psi\} \vdash \chi$.

The first clause gives us the general elimination form of *modus (ponendo) ponens*:

$$\dfrac{\phi \to \psi \qquad \phi \qquad \begin{matrix}[\psi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\; m \;\; \to\text{-elimination.}$$

The third clause too is straightforward in its import:

$$\dfrac{\psi \qquad \begin{matrix}[\phi \to \psi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\; m \;\; \to\text{-introduction (c).}$$

It's the second clause, and, in particular, what to do with that $\phi \in \Sigma_\infty$ in the consequent, that at first sight poses a problem. But we can think of it this way: the clause as a whole tells us that $\Sigma_\infty \cup \{\phi \to \psi\}$'s entailing χ suffices for ϕ's belonging to Σ_∞. And ϕ belongs to Σ_∞ if, and only if, the assumption that adding ϕ to Σ_∞ lets us derive χ is equivalent to saying that Σ_∞ itself entails χ. Putting this in the form of a rule, we get:

$$\cfrac{\begin{array}{cc} [\phi]^m & [\phi \to \psi]^m \\ \vdots & \vdots \\ \chi & \chi \end{array}}{\chi}\ m \ \to\text{-introduction (a).}$$[3]

Here's another way to think about this. If $\Sigma_\infty \cup \{\phi \to \psi\} \vdash \chi$ only if $\phi \in \Sigma_\infty$ then there's an incoherence in having both $\Sigma_\infty \cup \{\phi \to \psi\} \vdash \chi$ and $\Sigma_\infty \cup \{\phi\} \vdash \chi$, *i.e.*, $\phi \notin \Sigma_\infty$. Now, *in context*, '$\Sigma_\infty \vdash \chi$' is a way of expressing that incoherence, for our starting point is, exactly, that $\Sigma_\infty \nvdash \chi$.

These both help when we turn to negation, as we now do. We have that $\neg\phi \in \Sigma_\infty$ if, and only if, $\phi \notin \Sigma_\infty$. We have, on the one hand, that that $\neg\phi \in \Sigma_\infty$ and that $\phi \in \Sigma_\infty$ are jointly incoherent. On the first way, this gives us the familiar $\neg$-elimination rule, *ex falso quodlibet*:

$$\frac{\phi \qquad \neg\phi}{\chi}\ \neg\text{-elimination.}$$

We have, on the other hand, that that $\neg\phi \notin \Sigma_\infty$ and that $\phi \notin \Sigma_\infty$ are jointly incoherent. On the second way, this gives us the Rule of Dilemma as $\neg$-introduction rule:

$$\cfrac{\begin{array}{cc} [\phi]^m & [\neg\phi]^m \\ \vdots & \vdots \\ \chi & \chi \end{array}}{\chi}\ m \ \neg\text{-introduction.}$$

Before we turn our attention to rules for the quantifiers, let's look at these propositional logic rules a little more closely. We've read the rules off the closure properties of a set of formulas true (satisfied) in a model as classically understood. If the rules capture those properties, rather than just being in some way consequences of them, then, in another backwards journey, we should be able to read the semantic constraints off the rules. And so we can. We read them as follows: label categorically occurring subformulas as *true*, hypothetically as *false*; label formulas in which the connective of interest occurs the other way around—hypothetical = *true*, categorical = *false*. Doing this we revisit the utterly familiar:

- the $\wedge$-elimination (l) rule tells us that $\phi \wedge \psi$ is false when ϕ is false;

[3]Elsewhere I have called this 'Tarski's Rule' (Milne, 2008, 2010) for it bears the same relation to the tautology sometimes called 'Tarski's Law' as the better known Peirce's Rule does to Peirce's Law.

- the $\wedge$-elimination (r) rule tells us that $\phi \wedge \psi$ is false when ψ is false;
- the $\wedge$-introduction rule tells us that $\phi \wedge \psi$ is true when ϕ and ψ are both true;
- the $\vee$-elimination rule tells us that $\phi \vee \psi$ is false when ϕ and ψ are both false;
- the $\vee$-introduction (l) rule tells us that $\phi \vee \psi$ is true when ϕ is true;
- the $\vee$-introduction (r) rule tells us that $\phi \vee \psi$ is true when ψ is true;
- the $\rightarrow$-elimination rule tells us that $\phi \rightarrow \psi$ is false when ϕ is true and ψ is false;
- the $\rightarrow$-introduction (a) rule tells us that $\phi \rightarrow \psi$ is true when ϕ is false;
- the $\rightarrow$-introduction (c) rule tells us that $\phi \rightarrow \psi$ is true when ψ is true;
- the $\neg$-elimination rule tells us that $\neg\phi$ is false when ϕ is true;
- the $\neg$-introduction rule tells us that $\neg\phi$ is true when ϕ is false.

Now consider the binary connective with the truth-table (exclusive disjunction) in Table 1.

	$\phi + \psi$	ψ: t	ψ: f
ϕ	t	f	t
	f	t	f

Table 1: Truth-table for exclusive disjunction

We can read off two $+$-introduction rules. In the order top right, bottom left we get:

$$\dfrac{\phi \qquad \begin{matrix}[\psi]^m \\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\phi+\psi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\ m; \qquad \dfrac{\begin{matrix}[\phi]^m \\ \vdots \\ \chi\end{matrix} \qquad \psi \qquad \begin{matrix}[\phi+\psi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\ m.$$

And we can read off two $+$-elimination rules. In the order top left, bottom right we get:

$$\dfrac{\phi + \psi \qquad \begin{matrix}[\phi]^m \\ \vdots \\ \chi\end{matrix} \qquad \begin{matrix}[\psi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\, m; \qquad \dfrac{\phi + \psi \qquad \phi \qquad \psi}{\chi}\ .$$

We can go backward and forward between introduction and elimination rules and truth-tables. (And we can do this for connectives of any *arity*; see (Milne, 2015, §8.4.3) for more on this.)

We turn to the quantifiers and return to classical logic. We have that $\forall x\phi(x) \in \Sigma_\infty$ only if, for all terms t, $\phi(t) \in \Sigma_\infty$. Contraposing, $\forall x\phi(x) \notin \Sigma_\infty$ if, for some t, $\phi(t) \notin \Sigma_\infty$; and so, for any t, $\Sigma_\infty \cup \{\forall x\phi(x)\} \vdash \chi$ if $\Sigma_\infty \cup \{\phi(t)\} \vdash \chi$ which gives us the familiar elimination rule (recast in general elimination form):

$$\dfrac{\forall x\phi(x) \qquad \begin{matrix}[\phi(t)]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\, m\ \forall\text{-elimination.}$$

For the introduction rule for the universal quantifier—and the elimination rule for the existential quantifier—we *have* to take a more adventurous, and possibly less convincing, line; this is because parametric occurrences of names are not like ordinary individual constants—they really only exist in proofs, not in the model theory.[4] And tied in with that is the fact that we haven't really made provision for $\phi(a)$ occurring in the enumeration of formulas used when obtaining Σ_∞. Still, let's press on. (*Allez en avant, et la foi vous viendra!*) Somewhat in the style of Kit Fine's theory of arbitrary objects (Fine, 1985), we introduce a name 'a' which behaves syntactically as an individual constant but which we treat as though it were the name of a "generic", "representative" object. As such we attribute to a just those properties possessed by all elements in the (notional) domain. With that in place, we have that $\phi(a) \in \Sigma_\infty$ if, and only if $\forall x\phi(x) \in \Sigma_\infty$. Consequently, $\Sigma_\infty \cup \{\forall x\phi(x)\} \vdash \chi$ if $\Sigma_\infty \cup \{\phi(a)\} \vdash \chi$. How does '$a$''s status as name for a *generic representative* manifest itself?—In our making no specific assumptions about a. So we obtain the rule

[4] Put another way, although grammatically they are proper names in a language, they have no determinate reference (nor sense). For more on this, see (Milne, 2007, §§1 & 2).

$$\dfrac{\phi(a) \qquad \begin{matrix}[\forall x\phi(x)]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\; m\ \forall\text{-introduction}$$

where a does not occur in any premise upon which $\phi(a)$ depends.

In similar fashion we obtain the rules for the existential quantifier:

$$\dfrac{\phi(t) \qquad \begin{matrix}[\exists x\phi(x)]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\; m\ \exists\text{-introduction.}$$

and

$$\dfrac{\exists x\phi(x) \qquad \begin{matrix}[\phi(a)]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\; m\ \exists\text{-elimination}$$

where a occurs neither in χ nor in any side premise upon which χ depends.

These rules are all classically sound and, since we can obtain standard rules from them, they are complete. Moreover, the rules for $\wedge$, $\vee$, $\rightarrow$, $\neg$ and $\exists$ give us a classically complete system *with the subformula property* for the $\{\wedge, \vee, \rightarrow, \neg, \exists\}$-fragment of classical, first-order logic (Milne, 2010, §3.3; Sandqvist, 2012).

As one quick example, the intuitionistically invalid $(\phi \rightarrow \psi) \rightarrow \psi \vdash \phi \vee \psi$ can be derived like this:

$$\dfrac{\dfrac{[\phi]^4 \quad [\phi \vee \psi]^2}{\phi \vee \psi}\, 2\ \vee\text{-i} \qquad \dfrac{\dfrac{(\phi \rightarrow \psi) \rightarrow \psi \quad [\phi \rightarrow \psi]^4 \quad [\psi]^1}{\psi}\, 1\ \rightarrow\text{-e} \qquad [\phi \vee \psi]^3}{\phi \vee \psi}\, 3\ \vee\text{-i}}{\phi \vee \psi}\, 4\ \rightarrow\text{-i}\ .$$

Addendum At Hejnice, Melvin Fitting asked whether we get anything new if we apply the line of argument developed here to intuitionist logic rather than classical. I said then that I suspected that one gets nothing essentially new. That is indeed the case. Perhaps the conditional will be sufficient illustration.

In the canonical Kripke model, nodes are prime theories and we have that

$$\phi \to \psi \notin \Sigma_\infty \text{ iff, for some prime theory } \Delta \text{ such that } \Sigma_\infty \subseteq \Delta, \\ \phi \in \Delta \text{ and } \psi \notin \Delta.$$

On the one hand, as in the classical case, we have, then, that if $\Sigma_\infty \cup \{\psi\} \vdash \chi$ then $\Sigma_\infty \cup \{\phi, \phi \to \psi\} \vdash \chi$ and we get again the general elimination form of *modus (ponendo) ponens*. On the other hand, given the Lindenbaum–Henkin construction in the completeness proof for intuitionist logic, a prime theory Δ such that $\Sigma_\infty \subseteq \Delta$, $\phi \in \Delta$ and $\psi \notin \Delta$ exists if, and only if, $\Sigma_\infty \cup \{\phi\} \nvdash \psi$. That $\Sigma_\infty \cup \{\phi \to \psi\} \vdash \chi$ and that $\Sigma_\infty \cup \{\phi\} \vdash \psi$ are, then, incoherent constraints. This leads to the rule

$$\dfrac{\begin{matrix}[\phi]^m \\ \vdots \\ \psi\end{matrix} \qquad \begin{matrix}[\phi \to \psi]^m \\ \vdots \\ \chi\end{matrix}}{\chi}\ m \to.$$

which is nothing other than a rewriting of the standard $\to$-introduction rule.

3 An intriguing feature of classical logic

Our formulation of first-order classical logic does not have the subformula property. It does, however, satisfy this constraint: if $\Sigma \vdash \phi$ then there is a derivation of ϕ from Σ in which at most $\neg\phi$, subformulas of members of $\Sigma \cup \{\phi\}$, and negations of proper subformulas of members of $\Sigma \cup \{\phi\}$ occur (as follows from Theorem 8 of (Milne, 2010, p. 210). This fact hints at—only hints at, every so light-handedly—a rather different treatment of negation, a treatment very much against the grain in proof-theoretic semantics but one which allows a uniform treatment of rules. (So much against the grain that in Carrollian spirit one might say its stands the proof-theoretic semantics/inferentialist account of negation on its head.)

We could get ourselves a whole lot more rules by noting facts such as this: $\neg(\phi \wedge \psi) \in \Sigma_\infty$ if, and only if, $\neg\phi \in \Sigma_\infty$ or $\neg\psi \in \Sigma_\infty$. Recast in terms of the Lindenbaum–Henkin condition of non-membership, we obtain:

- if $\Sigma_\infty \cup \{\neg\phi\} \vdash \chi$ and $\Sigma_\infty \cup \{\neg\psi\} \vdash \chi$ then $\Sigma_\infty \cup \{\neg(\phi \wedge \psi)\} \vdash \chi$;
- if $\Sigma_\infty \cup \{\neg(\phi \wedge \psi)\} \vdash \chi$ then $\Sigma_\infty \cup \{\neg\phi\} \vdash \chi$;
- if $\Sigma_\infty \cup \{\neg(\phi \wedge \psi)\} \vdash \chi$ then $\Sigma_\infty \cup \{\neg\psi\} \vdash \chi$.

These give us the rules:

$$\dfrac{\neg\phi \qquad \begin{matrix}[\neg(\phi\wedge\psi)]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\; m\ \ \neg\wedge\text{-introduction (l)}; \qquad \dfrac{\neg\psi \qquad \begin{matrix}[\neg(\phi\wedge\psi)]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\; m\ \ \neg\wedge\text{-introduction (r)};$$

$$\dfrac{\neg(\phi\wedge\psi) \qquad \begin{matrix}[\neg\phi]^m\\ \vdots\\ \chi\end{matrix} \qquad \begin{matrix}[\neg\psi]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\; m\ \ \neg\wedge\text{-elimination.}$$

We can proceed similarly for disjunction, the conditional and the quantifiers, obtaining, for example,

$$\dfrac{\neg\phi(t) \qquad \begin{matrix}[\neg\forall x\phi(x)]^m\\ \vdots\\ \psi\end{matrix}}{\psi}\; m\ \ \neg\forall\text{-introduction.}$$

and

$$\dfrac{\neg\forall x\phi(x) \qquad \begin{matrix}[\neg\phi(a)]^m\\ \vdots\\ \psi\end{matrix}}{\psi}\; m\ \ \neg\forall\text{-elimination}$$

where a occurs neither in ψ nor in any premise other than $\neg\phi(a)$ upon which ψ depends.

Noting too that since $\neg\neg\phi \notin \Sigma_\infty$ if, and only if, $\phi \notin \Sigma_\infty$, we have the rules

$$\dfrac{\phi \qquad \begin{matrix}[\neg\neg\phi]^m\\ \vdots\\ \psi\end{matrix}}{\psi}\; m\ \ \neg\neg\text{-introduction}; \qquad \dfrac{\neg\neg\phi \qquad \begin{matrix}[\phi]^m\\ \vdots\\ \psi\end{matrix}}{\psi}\; m\ \ \neg\neg\text{-elimination.}$$

Albeit that they fall out of the Lindenbaum–Henkin conditions just as much as the rules above, these rules serve just as short cuts in a system with $\neg$-introduction and $\neg$-elimination. But these rules, together with the rules we already have for conjunction, disjunction, and the universal and existential quantifiers, *but not the rules for negation and the conditional*, all have a common structural feature: in the *introduction* rules, *all* side premises occur categorically; in the *elimination* rules, all side premises occur *hypothetically*.

And now, with negations in play, we can replace conditions such as $\phi \in \Sigma_\infty$ with $\neg\phi \notin \Sigma_\infty$, *i.e.*, $\Sigma_\infty \cup \{\neg\phi\} \vdash \chi$ to obtain rules of the same shapes:

$$\dfrac{\neg\phi \qquad \begin{matrix}[\phi\to\psi]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\; m\ \to\text{-i}, \qquad \dfrac{\psi \qquad \begin{matrix}[\phi\to\psi]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\; m\ \to\text{-i},$$

$$\dfrac{\phi\to\psi \qquad \phi \qquad \begin{matrix}[\psi]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\; m\ \to\text{-e};$$

$$\dfrac{\phi \qquad \neg\psi \qquad \begin{matrix}[\neg(\phi\to\psi)]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\; m\ \neg\to\text{-i},$$

$$\dfrac{\neg(\phi\to\psi) \qquad \begin{matrix}[\phi]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\; m\ \neg\to\text{-e}, \qquad \dfrac{\neg(\phi\to\psi) \qquad \begin{matrix}[\neg\psi]^m\\ \vdots\\ \chi\end{matrix}}{\chi}\; m\ \neg\to\text{-e}.$$

Now, rewrite *these* rules in the following ways.

Introduction rules Give each subformula ϕ_i a sequent to itself of the form $\Gamma_i \vdash \phi_i, \Delta_i$. These serve as the premises of right introduction rules. Give the formula introduced a sequent to itself of the form $\Gamma \vdash \star(\phi_1, \phi_2, \ldots), \Delta$ where Γ is the union of the antecedent side formulas Γ_i in premise sequents and Δ is the union of the succedent side formulas Δ_i in premise sequents. This serves as the conclusion of the rule.

Elimination rules Give each subformula ϕ_i a sequent to itself of the form $\Gamma, \phi_i \vdash \Delta_i$. These serve as the premises of left introduction rules. Give the formula eliminated a sequent to itself of the form $\Gamma, \star(\phi_1, \phi_2, \ldots) \vdash \Delta$ where, as before, Γ is the union of the antecedent side formulas in premise sequents and Δ is the union of the succedent side formulas in premises sequents. This serves as the conclusion of the rule.

For example, $\wedge$-introduction yields

$$\frac{\Gamma_1 \vdash \psi, \Delta_1 \qquad \Gamma_2 \vdash \phi, \Delta_2}{\Gamma_1, \Gamma_2 \vdash \psi \wedge \phi, \Delta_1, \Delta_2}\ \text{R } \wedge\text{-i},$$

$\wedge$-elimination gives us the pair of rules

$$\frac{\Gamma, \psi \vdash \Delta}{\Gamma, \psi \wedge \phi \vdash \Delta}\ , \qquad \frac{\Gamma, \phi \vdash \Delta}{\Gamma, \psi \wedge \phi \vdash \Delta}\ \text{L } \wedge\text{-i.}$$

$\neg\wedge$-introduction gives us the pair of rules

$$\frac{\Gamma \vdash \neg\psi, \Delta}{\Gamma \vdash \neg(\psi \wedge \phi), \Delta}\ , \qquad \frac{\Gamma \vdash \neg\phi, \Delta}{\Gamma \vdash \neg(\psi \wedge \phi), \Delta}\ \text{R } \neg\wedge\text{-i.}$$

$\neg\wedge$-elimination yields

$$\frac{\Gamma_1, \neg\psi \vdash \Delta_1 \qquad \Gamma_2, \neg\phi \vdash \Delta_2}{\Gamma_1, \Gamma_2, \neg(\psi \wedge \phi) \vdash \Delta_1, \Delta_2}\ \text{L } \neg\wedge\text{-i.}$$

The existential quantifier gives us

$$\frac{\Gamma, \psi(a) \vdash \Delta}{\Gamma, \exists x\psi \vdash \Delta}\ \text{L } \exists\text{-i;} \qquad \frac{\Gamma \vdash \psi(t), \Delta}{\Gamma \vdash \exists x\psi, \Delta}\ \text{R } \exists\text{-i}$$

where the parametric name a does not occur in any member of Γ or Δ.

The negated existential quantifier gives us

$$\frac{\Gamma, \neg\psi(t) \vdash \Delta}{\Gamma, \neg\exists x\psi \vdash \Delta}\ \text{L } \neg\exists\text{-i;} \qquad \frac{\Gamma \vdash \neg\psi(a), \Delta}{\Gamma \vdash \neg\exists x\psi, \Delta}\ \text{R } \neg\exists\text{-i}$$

where, again, the parametric name a does not occur in any member of Γ or Δ.

In these rules, by construction, the subformulas of interest and the formula containing the (left or right) introduced connective/quantifier all occur on the same side of the turnstile. Not so in the case of the standard sequent-calculus negation rules:

$$\frac{\Gamma \vdash \psi, \Delta}{\Gamma, \neg\psi \vdash \Delta}\ \text{L } \neg\text{-i;} \qquad \frac{\Gamma, \psi \vdash \Delta}{\Gamma \vdash \neg\psi, \Delta}\ \text{R } \neg\text{-i.}$$

But as induction on length of proof shows, any sequent derived using the left negation rule can be obtained without it when we include axioms of the form $\phi, \neg\phi \vdash$, where ϕ is atomic; likewise, any sequent derived using the right negation rule can be obtained without it when we include axioms of the form $\vdash \phi, \neg\phi$, where ϕ is atomic.

With axioms of the forms $\phi \vdash \phi$ and $\neg\phi \vdash \neg\phi$, ϕ atomic, what we have here is the *cut-free* sequent calculus $\mathscr{S}'$ of (Smullyan, 1968). Smullyan (1968, Ch. XV, §1) uses this calculus in proving first the Craig then the Craig–Lyndon interpolation theorem for first-order logic (without identity and function symbols). By attending very carefully to the steps in Smullyan's proof, we obtain a refinement of the interpolation theorem for classical first-order logic. The first step towards that refinement is to note that when we drop axioms of the form $\vdash \phi, \neg\phi$ we have a cut-free sequent calculus for the first-order variant of Kleene's strong three-valued logic (*e.g.*, van Benthem, 1988, Avron, 1991, 2003, Busch, 1993; the second, to note that when, instead, we drop axioms of the form $\phi, \neg\phi \vdash$ we have a cut-free sequent calculus for Graham Priest's Logic of Paradox (Avron, 1991, 2003).

Exactly because of what we noted about where the formulas of interest stand with respect to the turnstile, we have immediately that $\Gamma \vdash$ iff $\Gamma \vdash_{K3}$ and $\vdash \Delta$ iff $\vdash_{LP} \Delta$. And if $\Gamma \vdash \Delta, \Gamma \nvdash$ and $\nvdash \Delta$ then at least one axiom, either of the form $\phi \vdash \phi$ or of the form $\neg\phi \vdash \neg\phi$, must be used. In a derivation of $\Gamma \vdash \Delta$, we associate ϕ with the axiom $\phi \vdash \phi$, $\neg\phi$ with the axiom $\neg\phi \vdash \neg\phi$; we associate nothing with axioms of the forms $\phi, \neg\phi \vdash$ and $\vdash \phi, \neg\phi$. We then push the interpolants downwards, changing them as appropriate given the rule employed. The aim is to associate interpolants with all and only those sequents $\Gamma' \vdash \Delta'$ in the derivation such that $\Gamma' \nvdash$ and $\nvdash \Delta'$.

For example, the interpolant, if there is one, is unchanged by L $\wedge$-i, R $\vee$-i, L $\neg\vee$-i, R $\neg\wedge$-e, L $\neg\neg$-i, R $\neg\neg$-i, L $\forall$-i, R $\exists$-i, L $\neg\exists$-i, R $\neg\forall$, and by Weakening/Augmentation, and none is introduced if there isn't one associated with the premise sequent.

The other rules give rise to less straightforward stipulations. I'm not going to go through all of them. Here are two:

R $\wedge$-i Let η be such that $\Gamma_1 \vdash_{K3} \eta$ and $\eta \vdash_{LP} \phi, \Delta_1$ and only non-logical vocabulary common to both Γ_1 and $\Delta_1 \cup \{\phi\}$ occurs in η; let θ be such that $\Gamma_2 \vdash_{K3} \theta$ and $\theta \vdash_{LP} \psi, \Delta_2$ and only non-logical vocabulary common to both Γ_2 and $\Delta_2 \cup \{\psi\}$ occurs in θ. By R $\wedge$-i, $\Gamma_1, \Gamma_2 \vdash_{K3} \eta \wedge \theta$. By R $\wedge$-i, $\eta, \theta \vdash_{LP} \phi \wedge \psi, \Delta_1, \Delta_2$; by two applications of L $\wedge$-i, $\eta \wedge \theta \vdash_{LP} \phi \wedge \psi, \Delta_1, \Delta_2$. And only non-logical vocabulary common to both $\Gamma_1 \cup \Gamma_2$ and $\Delta_1 \cup \Delta_2 \cup \{\phi \wedge \psi\}$ occurs in $\eta \wedge \theta$.

If no interpolant is associated with $\Gamma_1 \vdash \phi, \Delta_1$ but $\Gamma_2 \vdash_{K3} \theta$ and $\theta \vdash_{LP}$

ψ, Δ_2 and only non-logical vocabulary common to both Γ_2 and $\Delta_2 \cup \{\psi\}$ occurs in θ, then, (*i*) if $\Gamma_1 \vdash$, no interpolant is associated with $\Gamma_1, \Gamma_2 \vdash \phi \wedge \psi, \Delta_1, \Delta_2$ and $\Gamma_1, \Gamma_2 \vdash$ by Weakening and (*ii*) if $\vdash \phi, \Delta_1$ then θ is taken as the interpolant, for we have both that $\Gamma_1, \Gamma_2 \vdash_{K3} \theta$ and that $\theta \vdash_{LP} \phi \wedge \psi, \Delta_1, \Delta_2$. Likewise, *mutatis mutandis*, if no interpolant is associated with $\Gamma_2 \vdash \psi, \Delta_2$ but there is one associated with $\Gamma_1 \vdash \phi, \Delta_1$.

If no interpolant is associated with $\Gamma_1 \vdash \phi, \Delta_1$ and none with $\Gamma_2 \vdash \psi, \Delta_2$, none is associated with $\Gamma_1, \Gamma_2 \vdash \phi \wedge \psi, \Delta_1, \Delta_2$. If $\Gamma_i \vdash$ then $\Gamma_1, \Gamma_2 \vdash$ by Weakening, $i = 1, 2$. If $\vdash \phi, \Delta_1$ and $\vdash \psi, \Delta_2$ then $\vdash \phi \wedge \psi, \Delta_1, \Delta_2$ by R $\wedge$-i.

L $\neg\forall$-i Let $\Gamma, \neg\phi(a) \vdash_{K3} \eta$ and $\eta \vdash_{LP} \Delta$ where the parameter a does not occur in any member of Γ or Δ and only non-logical vocabulary common to both $\Gamma \cup \{\neg\phi(a)\}$ and Δ occurs in η. The parameter a does not occur in η hence, by L $\neg\forall$-i, $\Gamma, \neg\forall x\phi \vdash_{K3} \eta$ and $\eta \vdash_{LP} \Delta$.

If no interpolant is associated with the sequent $\Gamma, \neg\phi(a) \vdash \Delta$, where the parameter a does not occur in any member of Γ or Δ, then none is associated with $\Gamma, \neg\forall x\phi \vdash \Delta$. If $\Gamma, \neg\phi(a) \vdash$ then, by L $\neg\forall$-i, $\Gamma, \neg\forall x\phi \vdash$ (as the parameter a does not occur in any member of Γ).

The difference between what I do and what Smullyan did is that where I associate nothing with axioms of the form $\phi, \neg\phi \vdash$ and $\vdash \phi, \neg\phi$, he associates, respectively, the constants **f** and **t**. This smooths out his stipulations for the awkward rules. As indicated by the above, I have to be quite careful with what I stipulate for some of the connectives, quantifiers, and combinations of these with negation.

Matters are so set up that if an interpolant ϕ is associated with a sequent $\Gamma \vdash \Delta$ then ϕ can be derived from Γ *without appeal to an axiom of the form* $\vdash \psi, \neg\psi$ and Δ can be derived from ϕ *without appeal to an axiom of the form* $\psi, \neg\psi \vdash$. In other words, with '$\vdash$' as classical consequence, if $\Gamma \vdash \Delta$, then

(i) $\Gamma \vdash_{K3}$, or

(ii) $\vdash_{LP} \Delta$, or

(iii) $\Gamma \nvdash$, $\nvdash \Delta$, and there's an interpolant ϕ such that $\Gamma \vdash_{K3} \phi$ and $\phi \vdash_{LP} \Delta$

where being an interpolant means that ϕ contains only non-logical vocabulary common to Γ and Δ. (Lyndon's parity constraints are also satisfied.

In (Milne, in press-b) I take a related approach: I use block tableaux and extend the result to classical first-order logic with identity (but not function symbols). In (Milne, in press-a) I give a semantic proof of the analogous result for classical propositional logic.)

We have that in classical logic, without identity and function-symbols, derivations can be limited to three kinds: those in which the premises are classically and so $K3$-inconsistent, those in which the conclusion is classically and so LP-logically true, and the rest in which there is an interpolant such that the first part of the derivation comprises a $K3$ derivation from antecedent to interpolant and the second part comprises an LP derivation from interpolant to succedent.

The use of the negated rules is essential to the line of argument here but the result stands however one has formulated classical logic (without identity and function-symbols). However, neither $K3$ nor LP has a straightforward natural deduction formulation so this fact about classically valid sequents tends to be lost from view.

4 A contraction-free approximation to classical logic

I really did come at what I'm about to describe backwards—completely backwards. This section owes its origin to Francesco Tonci Ottieri, a student in my undergraduate logic class at the University of Stirling in the Spring Semester in 2015. He came to see me one day with a diagrammatic account of the semantics of indicative conditionals. It took us—him and me, independently—a while to realise that the diagrammatic aspect was but one way of assigning values to formulas subject to the sole constraint

Schema 1 $v(\phi \to \psi) = v(\phi) - v(\psi).$

Under Schema 1, $v(\phi \to \psi) + v(\psi \to \chi) = v(\phi \to \chi)$. We will follow Ottieri in taking this to vindicate *hypothetical syllogism*.[5]

We'll call this ***Schema 2***:

$$v(\phi \to \psi) = v(\psi) - v(\phi).$$

While Schema 1 and Schema 2 do just as good a job with hypothetical syllogism, Scheme 2, but not Scheme 1, gives us that $v(\phi) + v(\phi \to \psi) = v(\psi)$. This we take as vindicating *modus (ponendo) ponens*.

[5] Ottieri employs Schema 1 as the basis of his *algebraic demonstration* in (Ottieri, in press).

Should we want our conditionals to contrapose and, more exactly, more demandingly, should we want $\phi \rightarrow \psi$ and $\neg\psi \rightarrow \neg\phi$ to take the same value, both schemata give us

$$\text{for any formulas } \phi \text{ and } \psi,\ v(\phi) + v(\neg\phi) = v(\psi) + v(\neg\psi).$$

The *only* way for this to work out is that, for some constant value K and all formulas ϕ,

$$v(\neg\phi) = K - v(\phi).$$

This gives us, as a consequence, that $v(\neg\neg\phi) = v(\phi)$.

Introducing a (*falsum*) constant, $\bot$, to which K is assigned as value, Schema 2, but not Schema 1, then gives us

$$v(\neg\phi) = v(\phi \rightarrow \bot),$$

leading to the familiar identification of $\neg\phi$ with $\phi \rightarrow \bot$.

Both schemata give us that $v(\phi \rightarrow \phi) = 0$, suggesting that 0 has some special role to play.

Schema 2, but not Schema 1, gives us this:

$$v(\phi_1) + v(\phi_2) + \ldots + v(\phi_n) + v(\psi) \gtreqless v(\chi)$$

(if, and) only if

$$v(\phi_1) + v(\phi_2) + \ldots + v(\phi_n) \gtreqless v(\psi \rightarrow \chi).$$

Once we sort out a little detail, we'll read this as vindicating Conditional Proof/$\rightarrow$-introduction.

Given what Schema 2 gets us, and what Schema 1 doesn't, we'll adopt Schema 2 as the preferred evaluation scheme for conditionals for the time being: for any formulas ϕ and ψ, $v(\phi \rightarrow \psi) = v(\psi) - v(\phi)$.

Classically, $\phi \wedge \psi$ is equivalent to $\neg(\phi \rightarrow \neg\psi)$. $v(\neg(\phi \rightarrow \neg\psi)) = K - ((K - v(\psi)) - v(\phi)) = v(\phi) + v(\psi)$.

Setting $v(\phi \wedge \psi) = v(\phi) + v(\psi)$, we have, for all formulas ϕ, that $v(\phi \wedge \neg\phi) = K$. On the model of hypothetical syllogism and *modus ponens* above, setting $v(\phi \wedge \psi) = v(\phi) + v(\psi)$ vindicates $\wedge$-introduction. We also have that $v(\phi \wedge \psi) \geq v(\phi)$ and that $v(\phi \wedge \psi) \geq v(\psi)$.

These last two suggest that we should take as our criterion of validity: the inference from ϕ_1, ϕ_2, ..., ϕ_n to ψ is valid, which we write as $\phi_1, \phi_2, \ldots, \phi_n \Vdash \psi$, if, and only if, under every valuation v,

$$v(\phi_1) + v(\phi_2) + \ldots + v(\phi_n) \geq v(\psi).^{6,7}$$

From this we get Gentzen's rule for $\rightarrow$-introduction: if $\phi_1, \phi_2, \ldots, \phi_m \Vdash \psi$ and $\chi_1, \chi_2, \ldots, \chi_n, \tau \Vdash \upsilon$ then $\phi_1, \phi_2, \ldots, \phi_m, \chi_1, \chi_2, \ldots, \chi_n, \psi \rightarrow \tau \Vdash \upsilon$, for if $v(\phi_1) + v(\phi_2) + \ldots + v(\phi_m) \geq v(\psi)$ and $v(\chi_1) + v(\chi_2) + \ldots v(\chi_n) + v(\tau) \geq v(\upsilon)$ then $v(\phi_1) + v(\phi_2) \ldots v(\phi_m) + v(\chi_1) + v(\chi_2) + \ldots v(\chi_n) + (v(\tau) - v(\psi)) \geq v(\upsilon)$.

Commutativity of addition gives us the structural rule Permutation (Exchange) in the antecedent.

Transitivity/Cut falls out, for if

$$v(\phi_1) + v(\phi_2) + \ldots + v(\phi_n) + v(\chi) \geq v(\tau)$$

and

$$v(\psi_1) + v(\psi_2) + \ldots + v(\psi_n) \geq v(\chi)$$

then—obviously!—

$$v(\phi_1) + v(\phi_2) + \ldots + v(\phi_n) + v(\psi_1) + v(\psi_2) + \ldots + v(\psi_n) \geq v(\tau).$$

Weakening/Augmentation/Monotonicity falls out *if we insist that formulas take only non-negative values*. This gives 0 its special status, as the minimum value, and since, then, for any formula ϕ, $K - v(\phi) = v(\neg\phi) \geq 0$, K is the maximum attainable value. We obtain the standard $\perp$-elimination rule, $\perp \Vdash \phi$, and *ex falso quodlibet* or *Explosion*, $\phi, \neg\phi \Vdash \psi$, for all formulas ϕ. (We could stipulate that $K = 1$ —but why bother? As long as $K > 0$ we get what we want and avoid trivialisation.)

From Augmentation *via* Conditional Proof we get the positive paradox of material implication: $\psi \Vdash \phi \rightarrow \psi$. Explosion and Conditional proof get us the negative paradox: $\neg\phi \Vdash \phi \rightarrow \psi$.

Restricting to non-negative values forces a small change in the way we treat conditionals and consequently conjunctions.

Schema 3 $v(\phi \rightarrow \psi) = \max\{v(\psi) - v(\phi), 0\}$.

[6]This might put the reader in mind, as it does me, of Dorothy Edgington's uncertainty semantics for classical logic augmented with Adams' conditionals(Edgington, 1992): there an inference is valid if under no probability distribution is the sum of the uncertainties of the premises less than the uncertainty of the conclusion (where *uncertainty* = 1 - *probability*).

[7]Had we put this criterion for validity in place sooner, we could have weakened contraposition so as to require only that, for all ϕ and ψ, $v(\phi \rightarrow \psi) \geq v(\neg\psi \rightarrow \neg\phi)$ and obtained the same stipulation for negation.

As a quick run through verifies, this makes no difference to any of the substantive claims made concerning conditionals under Schema 2 and validity and its preservation above. It does draw what we're doing closer to the evaluation clauses for connectives of Łukasiewicz's infinite-valued logic, especially as these are presented in, for example, Restall, 1994. The validity criterion is, however, non-standard and does give us something different.

If we use the classical equivalent $\neg\phi \to \psi$ to define (intensional) disjunction, we find that $\vee$-elimination fails (although obviously disjunctive syllogism is sound). On the other hand, under Schema 3, *but not Schema 2*, Łukasiewicz's definition of disjunction as $(\phi \to \psi) \to \psi$ gives us exactly the familiar (extensional) disjunction:

$$\begin{aligned} v((\phi \to \psi) \to \psi) &= \max\{v(\psi) - v(\phi \to \psi), 0\} \\ &= v(\psi) - v(\phi \to \psi) \\ &= v(\psi) - \max\{v(\psi) - v(\phi), 0\} \\ &= v(\phi) \text{ if } v(\phi) \leq v(\psi) \\ &= v(\psi) \text{ otherwise.} \end{aligned}$$

That is $v((\phi \to \psi) \to \psi) = \min\{v(\phi), v(\psi)\}$. We adopt this as our evaluation clause for disjunction. It vindicates the standard rules for disjunction.

What we end up with is a contraction-free "approximation" to classical propositional logic which includes the standard, intuitionist introduction and elimination rules for $\wedge$, $\vee$, $\to$ and $\neg$ and Double Negation Introduction and Double Negation Elimination; these rules, however, are read with *multisets*, not sets, of assumptions and those rules which permit discharge of an assumption in their application are restricted to allow discharge of *only one occurrence* of the assumption (including one in each branch in the case of $\vee$-elimination). It's obvious that

$$\phi \to (\phi \to \psi) \nvDash \phi \to \psi.$$

(And since we have taken what are usually thought of as intensional conjunction and extensional disjunction, we should expect some classical distributivity principles to fail—*e.g.*, $\phi \vee (\psi \wedge \chi) \nvDash (\phi \vee \psi) \wedge (\phi \vee \chi)$.)

The semantics can be given what, at first sight at least, seems a reasonably natural interpretation. We regard the numbers as measuring the "potential for introducing error" into our reasoning. Error permeates from occurrences of assumptions down to conclusions. *Each* time you use an assumption, you open up a channel through which error may permeate; discharging the assumption closes the channel. That, I think, is the picture. Steps of reasoning are vindicated if, of necessity, the conclusion has no greater potential for error than the sum of the error potentials of the assumptions on which it depends.[8,9]

5 Drawing the strands together

Our "ideal rules" are designed with proof of the completeness of classical first-order logic in mind. As far as that job is concerned, they are ideal. The elimination rules closely match Gentzen's. The introduction rules are different but ask yourself this: Where did Gentzen get his introduction rules from? He doesn't tell us but I think it's fair to say that he didn't pluck them out of thin air. He got them, I think, from intuitionism. Given the time and his interests, intuitionism was the obvious source for rules which are somehow supposed to display meaning in conditions for proof. But once we have the idea of reasoning with assumptions, we can look again at the role of rules of inference. There are two things one needs to know about logically complex assumptions: how to move on, having made one (elimination rules), and how to make do without them (introduction rules).

Our rules for the $\{\wedge, \vee, \rightarrow, \neg, \exists\}$-fragment do have the subformula property. This property doesn't carry over when we add the rules for the universal quantifier, the "constant domains inference" $\forall x(\phi(x) \vee \psi(a)) \vdash \forall x\phi(x) \vee \psi(a)$ being an obvious source of trouble. It might not be a great hardship to read "$\forall x$" as "$\neg\exists x\neg$" but it would be good to have a better understanding of what it is about the universal quantifier that leads the "ideal rules" approach into trouble—if trouble it is. Rather than see it as a problematic feature of classical logic, we can see it as a prompt to rethink how to incorporate negation. Our starting point, in the search for rules ideally suited to proving completeness, gives us a lead, as we saw.

[8] There's some loose analogy here with *aggregation of risk* considerations drawn up against multi-premise closure of knowledge under competently deduced consequence—see, *e.g.*, (Hawthorne, 2004, p. 47). As far as I am aware, epistemologists haven't brought such considerations to bear against Contraction; they worry about the risk accruing to belief in premises, not the *uses* of those premises in deductions.

[9] At Hejnice, Walter Carnielli, Libor Běhounek, and Chris Fermüller, I think, all pointed out to me that one can carry out this exercise in a weaker structure than a closed interval of the non-negative real numbers under the standard ordering with 0 as one end-point. But partly because it's how, set on course by Ottieri, I came across it and partly because it has, or at least seems to have, this natural reading, I prefer to stick with the way just outlined.

Those rules helped us see that classically valid first-order inferences divide into those with inconsistent premises, those with logically true conclusions, and the rest. *Very roughly*, when deriving one of the rest, you can proceed initially as though you don't care that truth and falsity are exhaustive, reach a "middle point", and proceed thereafter as though you do now care about that but not that truth and falsity are mutually exclusive.

Given a set Γ of formulas, let $\neg\Gamma$ be the set $\{\neg\phi : \phi \in \Gamma\}$. We have that

$$\Gamma \vdash_{K3} \neg\Delta \text{ if, and only if, } \Delta \vdash_{LP} \neg\Gamma.$$

So there's a sense in which the advocate of Kleene's three-valued logic can recapture every classically valid inference; and likewise the advocate of Priest's Logic of Paradox. But neither of these logics is proof-theoretically *nice*, at least not in any of the ways inferentialists have come to value. Having negation play a special role, as it does in Smullyan's calculus, may be a first step to addressing those misgivings but may also, I suppose, just be a case of swapping one mystery for another.

Gentzen's introduction and elimination rules, restricted by a "discharge policy" that allows formula-instances to be discharged only one at a time, where Gentzen allows any number of instances of the formula in question on which the intermediate conclusion depends to be discharged in an application of the rule, give us a logic weaker than classical. Restricted this way, the standard proof of the Law of Excluded Middle,

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\phi]^1}{\phi \lor \neg\phi}\;\lor\text{-i} \qquad [\neg(\phi \lor \neg\phi)]^2}{\bot}\;\bot\text{-i}}{\neg\phi}\;1\ \neg\text{-i}}{\phi \lor \neg\phi}\;\lor\text{-i} \qquad [\neg(\phi \lor \neg\phi)]^2}{\bot}\;\bot\text{-i}}{\neg\neg(\phi \lor \neg\phi)}\;2\ \neg\text{-i}}{\phi \lor \neg\phi}\;DNE,$$

is blocked when we try to discharge both occurrences of the assumption $\neg(\phi\lor\neg\phi)$ at the same time. We can turn this into a sound proof—sound with respect to the "error potential" semantics—of the conditional $\neg(\phi \lor \neg\phi) \rightarrow (\phi \lor \neg\phi)$ but that's not quite the same thing—really, *not at all* the same thing. The structural rule of Contraction, *i.e.*, the liberal discharge policy, is important. But now, here's a question: What makes a rule *structural*?

Section 1's "general introduction rules" $\neg$-introduction (Dilemma) and

$\rightarrow$-introduction (a) (Tarski's Rule) are not sound in the error potential semantics. This can be seen very straightforwardly because neither $\phi \vee \neg\phi$ nor $\phi \vee (\phi \rightarrow \psi)$ must take the value 0. If, within that framework, we wish to amend the evaluation clauses so as to make them sound, we are *forced* to two-valuedness. At this point, something remarkable happens. The validity criterion becomes equivalent to this:

> under every valuation the conclusion takes the value 0 when all the premises do.

With that being the case, occurrences of a formula greater than one in number in the premises cease to make a difference, *i.e.*, Contraction *is* now sound. For example, whereas we had $\phi, \phi \Vdash \phi \wedge \phi$ but *not* $\phi \Vdash \phi \wedge \phi$, we now have the latter. We can *derive* it as follows:

$$\dfrac{\phi \qquad \dfrac{\dfrac{\dfrac{\dfrac{[\phi]^1 \quad [\phi]^2}{\phi \wedge \phi}\,\wedge\text{-i}}{\phi \rightarrow (\phi \wedge \phi)}\,\rightarrow\text{-i (c)} \qquad [\phi \rightarrow (\phi \wedge \phi)]^1}{\phi \rightarrow (\phi \wedge \phi)}\,1\ \rightarrow\text{-i (a)} \qquad [\phi \rightarrow (\phi \wedge \phi)]^2}{\phi \rightarrow (\phi \wedge \phi)}\,2\ \rightarrow\text{-i (a)}}{\phi \wedge \phi}\,\rightarrow\text{-e}$$

At the cost of failure of the subformula property, we can work around retention of the illiberal discharge policy—a structural feature—by exploiting what was introduced as an operational rule.

Do general introduction rules blur the boundary between the operational and the structural? I suspect that they do. If so, is that a bad thing? It's tempting to think that the answer must be yes but I'm not sure why.

* * *

I have to say that I find these explorations on the other side of the mirror intriguing. I rather think they must have implications for inferentialism regarding the meaning of logical connectives and quantifiers—proof-theoretic semantics as the topic is often called. I have to say too, though, that I am at present far from clear as to what exactly those implications might be. Ending on that note, I'll take my cue from Lewis Carroll:

The Red Queen shook her head. "You may call it 'nonsense' if you like," she said, "but *I've* heard nonsense, compared to which that would be as sensible as a dictionary!"

References

Avron, A. (1991). Natural 3-Valued Logics—Characterization and Proof Theory. *Journal of Symbolic Logic*, *56*, 276-294.

Avron, A. (2003). Classical Gentzen-type Methods in Propositional Many-Valued Logics. In M. Fitting & E. Orlowska (Eds.), *Beyond Two: Theory and Applications of Multiple-Valued Logic* (p. 117-55). Heidelberg: Physica Verlag.

Busch, D. R. (1993). A Sequent Axiomatization of Three-valued Logic with Two Negations. In L. M. Pereira & A. Nerode (Eds.), *Logic Programming and Non-monotonic Reasoning, Proceedings of the Second International Workshop, Lisbon, Portugal, June 1993* (p. 476-494). Cambridge MA: MIT Press.

Carroll, L. (1970). *The Annotated Alice* (revised ed.; M. Gardner, Ed.). Harmondsworth: Penguin.

Edgington, D. (1992). Validity, Uncertainty and Vagueness. *Analysis*, *52*, 193-204.

Fine, K. (1985). *Reasoning with Arbitrary Objects*. Basil Blackwell.

Hawthorne, J. (2004). *Knowledge and Lotteries*. Oxford: Oxford University Press.

Milne, P. (2007). Existence, Freedom, Identity, and the Logic of Abstractionist Realism. *Mind*, *116*, 23-53.

Milne, P. (2008). A formulation of first-order classical logic in natural deduction with the subformula property. In M. Peliš (Ed.), *Logica Yearbook 2007* (p. 97-110). Prague: Filosofia.

Milne, P. (2010). Subformula and separation properties in natural deduction *via* small Kripke models. *Review of Symbolic Logic*, *3*, 175-227.

Milne, P. (2015). Inversion principles and introduction rules. In H. Wansing (Ed.), *Dag Prawitz on Proofs and Meaning* (p. 189-224). Cham, Heidelberg, New York, Dordrecht, London: Springer.

Milne, P. (in press-a). A non-classical refinement of the interpolation property for classical propositional logic. *Logique et Analyse*.

Milne, P. (in press-b). A refinement of the Craig–Lyndon interpolation theorem for classical first-order logic (with identity). *Logique et Analyse*.

Næss, A. (1968). *Four Modern Philosophers*. Chicago & London: University of Chicago Press. (English translation by Alastair Hannay of Norwegian original.)

Ottieri, F. T. (in press). Logica, Ordine Geometrico Demonstrata. *British Journal of Undergraduate Philosophy*.

Restall, G. (1994). *Arithmetic and Truth in Łukasiewicz's Infinitely Valued Logic* (Tech. Rep. No. TR-ARP-6-94). Australian National University, Canberra: Automated Reasoning Project.

Sandqvist, T. (2012). The subformula property in natural deduction established constructively. *Review of Symbolic Logic*, *5*, 710-719.

Smullyan, R. M. (1968). *First-Order Logic*. New York: Springer. (Reprinted with corrections and a new preface, New York: Dover Publications, 1995.)

van Benthem, J. (1988). *A Manual of Intensional Logic* (second ed.). Stanford: CSLI Publications.

Peter Milne
University of Stirling
Scotland
E-mail: peter.milne@stir.ac.uk