CONTRIBUTIONS TO THE THEORY OF 0-SIMPLE INVERSE SEMIGROUPS

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Introduction

The structure of 0-bisimple and bisimple inverse semigroups has been extensively studied and established by Clifford, Reilly, Warne, Munn and McAlister. The initial work was done by Clifford in [0] on bisimple inverse semigroups with an identity and this was generalised by Reilly and Clifford in [13]. In [5] McAlister has produced a structure theorem for 0-bisimple inverse semigroups in terms of groups and semilattices which can be specialised to give most of the previously known results in this area. These include the result of Munn described in [10] and the result of Reilly in [12], which deals with bisimple inverse semigroups whose semilattices are order isomorphic to the non negative integers with the reverse of the natural ordering, i.e. semilattices which are ω-chains. Warne has made a study of those bisimple inverse semigroups whose semilattices are order isomorphic to the integers with the reverse of the natural ordering and has obtained in [14] a structure for these which ties closely with [12].

To date, the corresponding work on 0-simple and simple inverse semigroups is more scanty, although specific types of simple inverse semigroups have been tackled. Munn in [7] has produced a structure theorem for simple inverse semigroups, whose semilattices are ω-chains, which is formulated in terms of groups and homomorphisms. An equivalent form of this result was obtained independently by Kochin in [2]. The result of Munn described above was generalised by Lallement, in [3] and by Petrich and Ault [15], to the case of 0-simple inverse semigroups whose semilattices are 0-direct unions of ω-chains with zero. In [11] Munn presents a structure theorem for a simple inverse semigroup S with a semilattice E of the following type: there exists a semilattice Y, with a greatest element, such that $E = N \times Y$, where N denotes the non-negative integers; the ordering on E is given by

$$(i,\alpha) \leq (j,\beta) \iff (i = j \text{ and } \alpha \leq \beta) \text{ or } i > j$$

and, moreover, the factorisation of E is compatible with the $\mathcal{D}$-structure of S in the sense that

$$((i,\alpha),(j,\beta)) \in \mathcal{D} \iff \alpha = \beta$$

It is shown that S is isomorphic to a semigroup of the form $N \times A \times N$, with a suitably defined multiplication, where A is a semilattice of groups with semilattice Y. This is a generalisation of the form $N \times G \times N$ described by Reilly in [12]. If the case that Y is a finite chain is considered we have the result obtained by Munn in [7] and when Y consists of a single element we are in the situation of [12].

In this thesis an attempt is made to generalise firstly the results obtained in [3] and secondly those obtained in [11]. For the first of these we aim at establishing a structure theorem for a 0-simple inverse semigroup whose semilattice is such that every non zero principal ideal is an ω-chain with zero, a type of semilattice described as an ω-tree with zero. A construction is developed using as a model the construction used by Munn in [7] and employing results established in the text regarding 0-simple inverse semigroups, whose semilattices are ω-trees with zero, which have no non trivial congruences contained in the Green's relation $\mathcal{H}$, (the "fundamental" 0-simple inverse semigroups whose semilattices are ω-trees with zero). The construction, involving an ω-tree with zero and a finite set of groups and homomorphisms, is shown to produce a 0-simple inverse semigroup whose semilattice is an ω-tree with zero and then, conversely, every 0-simple inverse semigroup whose semilattice is an ω-tree with zero is shown to have the form of the constructed semigroup.

Two main routes of specialisation of this result are possible. The first is to consider particular types of ω-trees with zero and the second is to consider the 0-bisimple inverse semigroups of this type. Starting with the semilattice, one of the first results obtained is one concerning semigroups of the same type as those considered by Lallement in [3]. Lallement's result and that in section 2.6 although differently formulated can be shown to be equivalent. A next step could be the consideration of a semilattice which is an ω-chain with zero, a situation which is exactly that of [7] with a zero adjoined. We can of course consider an ω-tree, instead of an ω-tree with zero, and from this obtain the structure of a simple inverse semigroup whose semilattice is an ω-tree. A simplification of this is to consider a semilattice which is order isomorphic to the integers with the reverse of the natural ordering. If simultaneously we restrict the number of groups under consideration to one, we have the situation of Warne in [14]. This reduction of the number of groups to one is based on a result obtained in Chapter 2 which states that the number of groups involved and the number of non zero $\mathcal{D}$-classes of the semigroup are equal. Clearly this leads us to consider 0-bisimple inverse semigroups whose semilattices are ω-trees with zero. From this we can obtain the structure of these exactly as in [5] and can deduce the result of [12].

The second type of semigroup considered is a 0-simple inverse semigroup S whose semilattice E is said to admit a factorisation compatible with the $\mathcal{D}$-structure of S. This is a development of the notion introduced by Munn in [11] and described above. We require that $E \setminus \{0\} = (F \setminus \{0\}) \times Y$ where F is a semilattice with zero and Y is a semilattice with greatest element; the ordering on E is given by

$$0 < (x,\alpha) \quad \text{for all } x \in F\setminus\{0\},\ \alpha \in Y$$

$$(x,\alpha) \leq (y,\beta) \iff (x = y \text{ and } \alpha \leq \beta) \text{ or } x < y$$

and, moreover, this factorisation is compatible with the $\mathcal{D}$-structure of S in the sense that

$$((x,\alpha),(y,\beta)) \in \mathcal{D} \iff \alpha = \beta.$$

In Munn's formulation of the situation in [11] the semilattice F is an ω-chain. In the course of Chapter 3 it is shown that if a 0-simple inverse semigroup is such that its semilattice admits a factorisation compatible with the $\mathcal{D}$-structure of the semigroup then the semilattice F involved is 0-uniform.

The approach to this second problem is to generalise a particular version of McAlister's construction of [5]. We restrict the semilattices used in his construction to those which have an associative addition and replace the group used by a semigroup with an identity element. Conditions are established for the semigroup formed to be 0-simple and inverse. In the particular case that the semigroup with identity is a ~~union~~ semilattice of groups it is found that the constructed semigroup is 0-simple and inverse and its semilattice admits a factorisation compatible with the $\mathcal{D}$-structure of the semigroup. A converse result is obtained in the case of a 0-simple inverse semigroup whose semilattice admits a factorisation compatible with the $\mathcal{D}$-structure of the semigroup and which is such that the 'first factor' has a non zero principal ideal whose group of order automorphisms is trivial.

The special cases arising from this result are obtained by one of two methods: the first is to specialise the semilattice used in the construction and the second is to specialise the semigroup with identity either to a finite chain of groups or to one group. The first specialisation is that of the semilattice to an ω-tree with zero. If at this point the semigroup with identity is replaced by a group we have a 0-bisimple inverse semigroup whose semilattice is an ω-tree with zero which is described in [5] and which also arose as a special case of the theorems of Chapter 2. Taking the semilattice to be an ω-chain with zero gives rise to exactly the situation of [11] with a zero adjoined. If now the semigroup is taken to be a finite chain of groups we obtain the result of [7] and [2]. If lastly this finite chain of groups is shrunk to one group we have, again, the result of [12].

1. Preliminaries

The notation and basic definitions are as in Clifford and Preston [1]. Throughout N will be used to denote the set $\{0,1,2...\}$.

1.1 An inverse semigroup and its semilattice

1.1.1 Let S be a set and $\times$ a binary operation on S. Then $(S,\times)$ is a semigroup if, for all $a,b,c \in S$, $(a \times b) \times c = a \times (b \times c)$, i.e. $\times$ is an associative binary operation on S. Usually the $\times$ is omitted and ab written for $a \times b$ and we refer to the semigroup S rather than to $(S,\times)$ when there is no ambiguity.

1.1.2 Let S be a semigroup and let T be a subset of S. Then T is said to be a subsemigroup of S if, for all $a,b \in T$, $ab \in T$.

We now introduce two very common types of semigroup following as in (1.1.1) and (1.1.2) the definitions in [1].

1.1.3 A semigroup S is said to be regular if, for each element $a \in S$ there exists an element $x \in S$ such that $a = axa$. Those elements $x \in S$ such that $a = axa$ and $x = xax$ are called inverses of a. From [1, Lemma 1.14] we have that if S is regular then each element has at least one inverse. For if $a \in S$ and $x \in S$ and is such that $a = axa$ then we consider xax. We have $a(xax)a = (axa)xa = axa = a$ and $(xax)a(xax) = x(axa)xax = x(axa)x = xax$. Hence xax is an inverse of a.

This leads us to the next definition.

1.1.4 A semigroup S is said to be inverse if it is regular and each element has exactly one inverse. By [1, Theorem 1.17] an inverse semigroup can also be characterised as a regular semigroup in which any two idempotents commute. If S is an inverse semigroup and

$a \in S$, it is customary to denote by a^{-1} the unique inverse of a.

1.1.5 Turning now to the set of idempotents of a semigroup, we denote by E_S the set of idempotents of the semigroup S. A partial ordering can be defined on E_S by the rule that $e \leq f \iff e = ef = fe$.

1.1.6 A commutative semigroup of idempotents is called a semilattice. In a semilattice any pair of elements has a greatest lower bound with respect to the ordering defined above, the greatest lower bound of two elements being their product.

1.1.7 Returning to the case where S is an inverse semigroup we can readily show that E_S is a semilattice. If $e, f \in E_S$ then $ef = fe$ from (1.1.4) and so $(ef)(ef) = e(fe)f = e(ef)f = ef$ so that E_S is a subsemigroup of S. Clearly E_S is commutative and so E_S is a semilattice.

1.1.8 A particular type of semilattice which will occur frequently in the following sections is an ω-chain. An ω-chain is a semilattice of the form $\{e_i : i \in N, \text{ with } e_i > e_j \iff i < j\}$.

1.2 The Green's relations on a semigroup

1.2.1 Let S be a semigroup. We adopt the convention of [1, Section 1.1] that $S^1 = S$ if S has an identity element and that $S^1 = S$ with an identity adjoined otherwise.

1.2.2 The equivalence relations $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ are defined on S as follows:-

$$(a,b) \in \mathcal{L} \iff S^1a = S^1b$$

$$(a,b) \in \mathcal{R} \iff aS^1 = bS^1$$

$$(a,b) \in \mathcal{J} \iff S^1aS^1 = S^1bS^1$$

Clearly if a ≠ b then $(a,b) \in \mathcal{L}$ if and only if there exist $c,d \in S$ such that $ca = b$ and $a = db$. A similar result holds for $\mathcal{R}$.

1.2.3 We denote by L_a (R_a, J_a) the $\mathcal{L}$- $(\mathcal{R}$-, $\mathcal{J}$-) class of S containing a.

1.2.4 In [1, Lemma 2.1] it is shown that $\mathcal{L}$ and $\mathcal{R}$ commute and their product is defined to be $\mathcal{D}$. Clearly $(a,b) \in \mathcal{D}$ if and only if there exists $c \in S$ such that $(a,c) \in \mathcal{R}$ and $(c,b) \in \mathcal{L}$ (or alternately $(a,c) \in \mathcal{L}$ and $(c,b) \in \mathcal{R}$). The $\mathcal{D}$-class of S containing a is written D_a.

1.2.5 Finally the equivalence relation $\mathcal{H}$ on S is defined to be $\mathcal{L} \cap \mathcal{R}$. Thus $(a,b) \in \mathcal{H}$ if and only if $(a,b) \in \mathcal{L}$ and $(a,b) \in \mathcal{R}$. We denote the $\mathcal{H}$-class of S containing a by H_a.

These equivalence relations are known as the Green's relations , and are defined as in [1, Section 2.1].

1.2.6 In the case that S is an inverse semigroup we note that $a \in Sa$ and so $S^1a = Sa$. Similarly $aS^1 = aS$ and $S^1aS^1 = SaS$. This is proved in [1, Lemma 2.13]

1.2.7 Also, if S is inverse, we have from [1, Theorem 1.17] that each $\mathcal{R}$-class and each $\mathcal{L}$-class of S contain exactly one idempotent.

1.2.8 If we wish to emphasise the semigroup S on which the Green's relations are being discussed we write, for example, $\mathcal{L}_S$.

We now use the terminology of the Green's relations to make further descriptions of a semigroup as in [1, Section 2.1].

1.2.9 A semigroup S with a zero is said to be 0-bisimple if, for any pair $a,b \in S \setminus \{0\}$, $(a,b) \in \mathcal{D}$, i.e. there is one non zero $\mathcal{D}$-class in S. A semigroup S without zero is said to be bisimple if it consists of a single $\mathcal{D}$-class.

1.2.10 A semigroup S with a zero is said to be 0-simple if $S^2 \neq 0$ and for any pair $a,b \in S \setminus \{0\}$, $(a,b) \in \mathcal{J}$, i.e. there is one non zero $\mathcal{J}$-class in S. Also a semigroup S without zero is said to be simple if it consists of a single $\mathcal{J}$-class.

1.2.11 In [1, Section 1.7] the maximal subgroups of a semigroup S are defined to be those subgroups of S which are not properly contained in any other subgroup of S. From [1, Theorem 2.16] we have that if S is a semigroup then any $\mathcal{H}$-class of S containing an idempotent is a subgroup of S and indeed from [1, Section 2.3] the maximal subgroups of S are precisely the $\mathcal{H}$-classes of S containing idempotents.

1.2.12 If S is an inverse semigroup which is a union of groups then by [1, Section 4.2] S is a semilattice Y of groups where Y is isomorphic to E_S and, if $a \in S$, $H_a = L_a = R_a = D_a = J_a$. Thus we have that if S is an inverse semigroup which is a union of groups and if $a \in S$ then $aa^{-1} = a^{-1}a$ by (1.27) above. Conversely if S is an inverse semigroup in which $aa^{-1} = a^{-1}a$ for all $a \in S$ then, as $(a,aa^{-1}) \in \mathcal{R}$ and $(a,a^{-1}a) \in \mathcal{L}$, we have $(a,aa^{-1}) \in \mathcal{H}$. Hence a belongs to a maximal subgroup and we have that S is a union of maximal subgroups, i.e. a union of groups, and so is a semilattice of groups.

1.2.13 A semigroup with an identity element is called a monoid.

1.2.14 A centric inverse monoid is an inverse monoid which is a semilattice of groups.

1.3 Semigroup and semilattice related

In this section we introduce some special types of semigroup and semilattice and show the relations between them.

1.3.1 It is established in [6, Section 3] that, if S is an inverse semigroup, the maximum congruence contained in $\mathcal{H}$ is μ, where μ is defined as follows:- $(a,b) \in \mu \iff a^{-1}ea = b^{-1}eb$ for all $e \in E_S$. In the special case that $\mathcal{H}$ is a congruence $\mathcal{H} = \mu$.

1.3.2 According to [9, Section 2] if S is a semigroup in which the only congruence contained in $\mathcal{H}$ is the identity congruence i, then S is said to be fundamental. An inverse semigroup is fundamental if and only if $\mu = i$.

We now construct a fundamental inverse semigroup.

1.3.3 As in [1, Section 1.9] we define a one-to-one partial transformation on a set X to be a bijection with domain and range both subsets of X. The null mapping is also considered to be a partial transformation on X, being the mapping whose domain and range are both the empty set. If α is a partial transformation on X it is customary to denote by $\Delta(\alpha)$ the domain of α, and by $\nabla(\alpha)$ the range of α. The set of all one-to-one partial transformations on a set X is denoted by $\mathcal{I}_X$. A multiplication is defined on $\mathcal{I}_X$ as follows:- if $\alpha, \beta \in \mathcal{I}_X$ with $\nabla(\alpha) \cap \Delta(\beta) = \phi$ then $\alpha\beta = 0$, otherwise $\Delta(\alpha\beta) = (\nabla(\alpha) \cap \Delta(\beta))\alpha^{-1}$, $\nabla(\alpha\beta) = (\nabla(\alpha) \cap \Delta(\beta))\beta$ and if $x \in \Delta(\alpha\beta)$, then $x(\alpha\beta) = (x\alpha)\beta$.

It is shown in [1, Section 1.9] that $\mathcal{I}_X$ is an inverse semigroup.

1.3.4 Let E be a semilattice then T_E is defined in [6, Section 2] to be the subset of $\mathcal{I}_E$ comprising those partial transformations of E which have domain and range which are principal ideals of E. From [6, Lemma 2.2], T_E is an inverse subsemigroup of $\mathcal{I}_E$ and by [9, Corollary 2.7] is fundamental. As T_E is inverse we can apply [9, Lemma 1.2] to describe the Green's relations in T_E. We thus have that, in T_E, $(\alpha,\beta) \in \mathcal{L} \iff \nabla(\alpha) = \nabla(\beta)$ and $(\alpha,\beta) \in \mathcal{R} \iff \Delta(\alpha) = \Delta(\beta)$.

Next we follow the pattern of [9, Section 3] in making two definitions which relate E and T_E.

1.3.5 If E is a semilattice with zero we shall denote by E* the set $E \setminus \{0\}$.

1.3.6 Let E be a semilattice with zero. An inverse subsemigroup S of T_E is said to be 0-transitive if and only if it contains the zero of T_E, i.e. the mapping with domain and range {0}, and also, for all $e,f \in E^*$, there exists $\alpha \in S$ such that $\Delta(\alpha) = Ee$ and $\nabla(\alpha) = Ef$.

Let E be a semilattice without zero. Then the above definition is modified as follows. An inverse subsemigroup S of T_E is said to be transitive if and only if, for all $e,f \in E$, there exists $\alpha \in S$ such that $\Delta(\alpha) = Ee$ and $\nabla(\alpha) = Ef$.

1.3.7 Let E be a semilattice with zero then E is said to be 0-uniform if and only if, for all $e,f \in E^*$, $Ee \cong Ef$. If E is a semilattice without zero then E is said to be uniform if and only if, for all $e,f \in E$, $Ee \cong Ef$.

1.3.8 If we now examine these two definitions more closely we see that T_E is 0-transitive (transitive) if and only if it contains a 0-transitive (transitive) inverse subsemigroup. Also E is 0-uniform (uniform) if and only if T_E is 0-transitive (transitive).

Next we obtain the structure of T_E for a particular type of uniform semilattice E.

1.3.9 A partially ordered set P is said to be inversely well-ordered if every non-empty subset of P has a greatest element. This definition is given in [6, Section 3].

1.3.10 Let E be a uniform semilattice in which every principal ideal is inversely well-ordered. Clearly, as E is uniform, we have from [6, Theorem 2.3] that E is the semilattice of a bisimple semigroup. Let S denote this semigroup. We are now in the situation of [6, Theorem 3.2] and have that $\mathcal{H}$ is a congruence on S and $S/\mathcal{H} \cong T_E$. Hence $\mathcal{H} = i$ on T_E and so we can specify T_E completely. If $e,f \in E$, by the uniformity of E and the ensuing transitivity of T_E, there exists $\alpha \in T_E$ such that

$\Delta(\alpha) = Ee$ and $\nabla(\alpha) = Ef$. This element α is unique, as $\mathcal{H} = i$, by (1.3.4). Thus if we denote by $\xi_{e,f}$ the unique element of T_E with domain Ee and range Ef we have $T_E = \{\xi_{e,f} : e,f \in E\}$. Also $\Delta(\xi_{e,f}\,\xi_{g,h}) = (Ef \cap Eg)\,\xi_{e,f}^{-1} = (Efg)\,\xi_{e,f}^{-1} = E(fg\,\xi_{e,f}^{-1})$ by [6, Lemma 2.1] and $\nabla(\xi_{e,f}\,\xi_{g,h}) = E(fg\,\xi_{g,h})$, similarly. Thus

$$\xi_{e,f}\,\xi_{g,h} = \xi_{fg\xi_{e,f}^{-1},\,fg\,\xi_{g,h}}$$

1.3.11 It is easily seen that if E is a 0-uniform semilattice in which every principal ideal is inversely well-ordered then the result of (1.3.10) can be extended to include this situation. We would then have $T_E = \{\xi_{e,f} : e,f \in E^*\} \cup \{0\}$ with the non zero products in T_E being $\xi_{e,f}\,\xi_{g,h} = \xi_{fg\xi_{e,f}^{-1},\,fg\xi_{g,h}}$ where $fg \neq 0$. It is possible to regard T_E as $(E^* \times E^*) \cup \{0\}$ with multiplication defined on $(E^* \times E^*) \cup \{0\}$ as follows:-

$$(e,f)0 = 0(e,f) = 0^2 = 0$$

$$(e,f)(g,h) = \begin{cases} 0 \text{ if } fg = 0 \\ ((fg)\xi_{e,f}^{-1},\ (fg)\xi_{g,h}) \quad \text{if } fg \neq 0. \end{cases}$$

Finally in this section we include another two definitions relating E and T_E as in [9, Section 3].

1.3.12 Let E be a semilattice with zero. An inverse subsemigroup S of T_E is said to be 0-subtransitive if and only if S contains the zero of T_E and also, for all $e,f \in E^*$, there exists $\alpha \in S$ such that $\Delta(\alpha) = Ee$ and $\nabla(\alpha) \subseteq Ef$. If E is a semilattice without zero the definition is modified as follows. An inverse subsemigroup S of T_E is said to be subtransitive if and only if, for all $e,f \in E$, there exists $\alpha \in S$ such that $\Delta(\alpha) = Ee$ and $\nabla(\alpha) \subseteq Ef$.

1.3.13 A semilattice E with zero is said to be 0-subuniform if and only if, for all $e,f \in E^*$, there exists $g \leq f$ such that $Ee \cong Eg$. If E is a semilattice without zero then E is said to be subuniform if and only if, for all $e,f \in E$, there exists $g \leq f$ such that $Ee \cong Eg$.

1.3.14 We note that T_E is 0-subtransitive (subtransitive) if and only if it contains a 0-subtransitive (subtransitive) inverse subsemigroup. Also E is 0-subuniform (subuniform) if and only if T_E is 0-subtransitive (subtransitive).

2. A 0-simple inverse semigroup whose semilattice is an ω-tree with zero

In this chapter we show how to construct a 0-simple inverse semigroup whose semilattice is an ω-tree with zero from a finite set of groups and homomorphisms and an ω-tree with zero. Then we show that all 0-simple inverse semigroups with semilattice an ω-tree with zero are of this type.

2.1 An ω-tree with zero

2.1.1 A semilattice with zero in which every non zero principal ideal is an ω-chain with zero is called an ω-tree with zero.

2.1.2 Let E be an ω-tree with zero. If $e,f \in E^*$ with $e \geq f$ we define $[e,f] = |\{x \in E^*: e \geq x \geq f\}| - 1$. Since $\{x \in E^*: x \leq e\}$ is an ω-chain, this is a well-defined non-negative integer. For each $t \in N$ and $e \in E^*$ there exists a unique element $f \in E^*$ such that $f \leq e$ and $[e,f] = t$; we denote this element by $e + t$.

2.1.3 Define a relation $\sim$ on E^* by $a \sim b \iff ab \neq 0$. This relation is an equivalence relation. The equivalence classes of E^* generated by $\sim$ are called the components of E^*. Thus $a \sim b \iff$ a and b belong to the same component of E^*.

2.1.4 Select a tranversal T of the components of E^*. For each $a \in E^*$ let e_a denote the element of T in the same component as a. Let $k \in N$, with $k \geq 1$. For each element $a \in E^*$ we define the k-index of a, relative to T, to be the non negative remainder when $[e_a,ae_a] - [a,ae_a]$ is divided by k.

2.2 The semigroup $S\ (E, T, k)$

This section is an account of an unpublished result of W. D. Munn.

2.2.1 Let E be an ω-tree with zero and let T be a transversal of the components of E^*. Fix $k \in N$, with $k \geq 1$, and for all $a \in E^*$ let $\underline{a}$ denote the k-index of a, relative to T. Let $S\ (E, T, k) = \{(a,b) \in E^* \times E^*: \ \underline{a} = \underline{b}\} \cup \{(0,0)\}$. Define multiplication on $S\ (E, T, k)$ as follows:-

$$(a,b)(c,d) = (a + t,\ d + s) \text{ where } t = [b,bc] \text{ and}$$
$$s = [c,bc] \text{ if } bc \neq 0,$$

all other products are $(0,0)$.

2.2.2 We note that in the case that E is an ω-tree with zero then E is a 0-uniform semilattice in which every principal ideal is inversely well-ordered. Hence we can apply (1.3.11) and have $T_E = \{(e,f) \in E^* \times E^*\} \cup \{0\}$, with $(e,f)(g,h) = (fg\xi^{-1}_{e,f}, fg\xi_{g,h})$ if $fg \neq 0$ and all other products are zero. However $fg\xi^{-1}_{e,f} = e + n$ where $n = [f,fg]$ and $fg\xi_{g,h} = h + m$ where $m = [g,fg]$. From this we see that $S\ (E, T, k) \subseteq T_E$ and that the multiplication defined on $S\ (E, T, k)$ is that of T_E.

2.2.3 <u>Theorem:</u> The set $S\ (E, T, k)$ with the multiplication defined above is a 0-subtransitive inverse subsemigroup of T_E.

<u>Proof:</u> Let $(a,b), (c,d) \in S\ (E, T, k) \setminus \{(0,0)\}$. If $bc = 0$ then $(a,b)(c,d) = (0,0) \in S(E, T, k)$. Suppose therefore that $bc \neq 0$ and consider $(a,b)(c,d) = (a + t, d + s)$ where $t = [b,bc]$ and $s = [c,bc]$. To show that $(a + t, d + s) \in S\ (E, T, k)$ we have to prove that $\underline{a + t} = \underline{d + s}$. To simplify this we insert the following lemma.

2.2.4 Lemma: If $a \in E^*$ and $p \in N$, then $\underline{a + p} \equiv \underline{a} + p \pmod{k}$.

Proof: Since $a(a + p) \neq 0$ we have $e_a = e_{a + p}$. Also $[e_a,(a + p)e_a] - [a + p, (a + p)e_a] = [e_a, ae_a] + [ae_a,(a + p)e_a] - ([a,ae_a] + [ae_a,(a + p)e_a] - [a,a + p]) = [e_a,ae_a] - [a,ae_a] + [a,a + p] = [e_a,ae_a] - [a,ae_a] + p$. Hence $\underline{a + p} \equiv \underline{a} + p \pmod{k}$.

Returning to the theorem, we have from Lemma 2.2.4 that $\underline{a + t} \equiv \underline{a} + t \pmod{k}$. However $\underline{a} = \underline{b}$ and so $\underline{a + t} \equiv \underline{b} + t \pmod{k}$. Applying Lemma 2.2.4 again we have $\underline{b + t} \equiv \underline{b} + t \pmod{k}$ and so $\underline{a + t} = \underline{b + t}$. Similarly we have $\underline{d + s} = \underline{c + s}$. However $b + t = c + s = bc$ so that $\underline{a + t} = \underline{d + s}$. Thus the multiplication on $S (E, T, k)$ is closed and so $S (E, T, k)$ is a subsemigroup of T_E. It is an inverse subsemigroup of T_E since, if $(a,b) \in S (E, T, k)$, $(a,b)^{-1} = (b,a)$ is also in $S (E, T, k)$.

Let $a,b \in E^*$ and let $t = k - \underline{b} + \underline{a}$. Clearly $t \geq 1$ and so $b + t < b$. Also, by Lemma 2.2.4, we have $\underline{b + t} \equiv \underline{b} + t \pmod{k}$ and $\underline{b} + t = k + \underline{a} \equiv \underline{a} \pmod{k}$. Thus $(a,b + t) \in S (E, T, k)$ and so as $S (E, T, k)$ also contains the zero of T_E, S is 0-subtransitive.

The next theorem is the converse of Theorem 2.2.3.

2.2.5 Theorem: If S is a 0-subtransitive inverse subsemigroup of T_E, where E is an ω-tree with zero, then there exists $k \in N$, with $k \geq 1$, and T, a transversal of the components of E^*, such that S is of the form $S (E, T, k)$.

Proof: Since S is a 0-subtransitive inverse subsemigroup of T_E we have from (1.3.11) that S is a subset of $(E^* \times E^*) \cup \{(0,0)\}$ with multiplication as described in (1.3.11).

Fix $e \in E^*$ and define $k = \min \{ n \in N: n \geq 1 \text{ and } (e,e + n) \in S\}$ The 0-subtransitivity of S ensures the existence of such an integer. Select a transversal of the components of E^*. For all $a \in E^*$ let

Z_a be the element in the transversal such that $a \sim Z_a$. Since S is 0-subtransitive, for all $a \in E^*$ there exists $e_a \in E^*$ such that $e_a \leq Z_a$ and $(e, e_a) \in S$. Let $T = \{e_a : a \in E^*\}$. Then T is a transversal of the components of E*.

Before proceding with the remainder of the proof of this theorem it is convenient to consider the following lemmas.

2.2.6 <u>Lemma:</u> For all $p, q \in N$, $(e + p, e + q) \in S$ if and only if $p \equiv q \pmod{k}$.

<u>Proof:</u> We note firstly that by [9, Theorem 3.2 (ii)] S has a semi-lattice isomorphic to E.

Since $(e, e + k) \in S$, $(e, e + k)^n = (e, e + nk) \in S$ for all $n \in N$ with $n \geq 1$. Also $(e + i, e + i)(e, e + k)^n = (e + i, e + nk + i) \in S$, for all $n \in N$ with $n \geq 1$ and for all $i \in N$ with $0 \leq i < k$. As S contains the semilattice of T_{E^1} we also have $(e + i, e + i) \in S$ for all $i \in N$ and so we have $(e, e + nk) \in S$ for all $n \in N$ and $(e + i, e + i + nk) \in S$ for all $n \in N$ and for all $i \in N$ with $0 \leq i < k$.

Let $(e + p, e + q) \in S$ with $p = nk + i$ and $q = mk + j$ where $m, n, i, j \in N$ and $0 \leq i, j < k$. Then $(e + i, e + p) \in S$ and $(e + j, e + q) \in S$. Thus $(e + i, e + p)(e + p, e + q)(e + j, e + q)^{-1} \in S$ and so $(e + i, e + j) \in S$. Assume that $i > j$. Then $(e, e + k)(e + i, e + j) = (e, e + j + (k - i)) = (e, e + k - (i - j))$. However this is in S and $0 < k - (i - j) < k$ contradicting the definition of k. Hence $i \ngtr j$. Similarly we can show $i \nless j$ and so we have $i = j$ and $p \equiv q \pmod{k}$.

Conversely let $p, q \in N$ with $p \equiv q \pmod{k}$. Let $p = nk + i$ and $q = mk + i$ where $m, n, i \in N$ with $0 \leq i < k$. Then $(e + i, e + p) \in S$ and $(e + i, e + q) \in S$. So that $(e + i, e + p)^{-1}(e + i, e + q) = (e + p, e + q) \in S$.

2.2.7 Lemma. For all $a \in E^*$ and for all $p,q \in N$, $(a + p,\ a + q) \in S$ if and only if $p \equiv q \pmod{k}$.

Proof: For $a \in E^*$, define $k_a = \min\{\, n \in N : n \geq 1 \text{ and } (a,\ a + n) \in S \,\}$ with $k_e = k$. We then have, for all $a \in E^*$, a parallel result to Lemma 2.2.6, namely that $(a + p,\ a + q) \in S$ if and only if $p \equiv q \pmod{k_a}$. Since S is 0-subtransitive, for each $a \in S$ there exists $p \in N$ with $p \geq 1$ such that $(e,\ a + p) \in S$. Since $(e,\ e + k) \in S$ we have $(e,\ a + p)^{-1}\,(e,\ e + k)^{-1} = (a + p, e)(e + k, e) = (a + p + k, e)$ $\in S$. Hence we have $(a + p, e)(e, a + p + k) = (a + p,\ a + p + k)$ $\in S$. Thus, by the parallel result to Lemma 2.2.6, we have $p \equiv p + k \pmod{k_a}$ so that $k_a | k$. Also there exists $q \in N$, with $q \geq 1$, such that $(a,\ e + q) \in S$. Since $(a,\ a + k_a) \in S$ we have $(e + q,\ a)(a + k_a, a) = (e + q + k_a, a) \in S$. Thus $(e + q + k_a, a)(a, e + q) = (e + q + k_a,\ e + q) \in S$ and from Lemma 2.2.6 we now have $q + k_a \equiv q \pmod{k}$ and so $k | k_a$. Combining these two results we have $k = k_a$ and the lemma is proved.

Returning now to Theorem 2.2.5, let $(a,b) \in S \setminus \{(0,0)\}$. Since $ae_a \neq 0$ there exist $p,n \in N$ such that $a + p = e_a + nk$. By Lemma 2.2.4 $\underline{a + p} \equiv \underline{a} + p \pmod{k}$ so that $\underline{a} + p \equiv \underline{e_a + nk} \pmod{k}$. However $\underline{e_a + nk} = \underline{e_a} = 0$ and so $\underline{a} \equiv -p \pmod{k}$. Similarly there exist $q,m \in N$ such that $b + q = e_b + mk$ and $\underline{b} \equiv -q \pmod{k}$. We have, by Lemma 2.2.7, that $(e_a, e_a + nk),\ (e_b, e_b + mk) \in S$. By choice $(e, e_a),\ (e, e_b) \in S$ and so $(e_a + nk, e_a)(e_a, e)(e, e_b)(e_b, e_b + mk) \in S$ so that $(e_a + nk, e_b + mk) = (a + p,\ b + q) \in S$. Since $(a,b) \in S$ and S is inverse, $(b,a) \in S$ and we have $(b,a)(a + p,\ b + q) \in S$, i.e. $(b + p,\ b + q) \in S$. Thus, by Lemma 2.2.7, $p \equiv q \pmod{k}$ and so $\underline{a} = \underline{b}$. Hence we have $S \subsetneq S(E, T, k)$.

Now suppose that $(a,b) \in S(E, T, k)$ with $a,b \neq 0$. Then $\underline{a} = \underline{b}$. Since S is 0-subtransitive there exist $p,q \in N$ such that

$(a, e+p) \in S$ and $(b, e+q) \in S$. However, from above, $S \subseteq S(E, T, k)$ and so $\underline{a} = \underline{e+p}$ and $\underline{b} = \underline{e+q}$. Thus $\underline{e+p} = \underline{e+q}$. From Lemma 2.2.4 $\underline{e+p} \equiv \underline{e} + p \pmod{k}$ and $\underline{e+q} \equiv \underline{e} + q \pmod{k}$ and so we have $p \equiv q \pmod{k}$. Hence, by Lemma 2.2.6, $(e+p, e+q) \in S$. From this $(a, e+p)(e+p, e+q)(e+q, b) = (a,b) \in S$ and we have $S(E, T, k) \subseteq S$.

Combining the two inclusion results we have $S = S(E, T, k)$.

2.3 The construction of the groupoid $S(E, T, k, G_i, \gamma_i, e, v_f)$

In this section we describe a process for constructing a 0-simple inverse semigroup from a finite set of groups and homomorphisms and an ω-tree with zero.

2.3.1 Let E be an ω-tree with zero and let T be a transversal of the components of E^*. Fix $k \in N$, with $k \geq 1$, and for all $a \in E^*$ let $\underline{a}$ be the k-index of a, relative to T.

2.3.2 Let $G_0, G_1, \ldots, G_{k-1}$ be k groups with identity elements $e_0, e_1 \ldots, e_{k-1}$ respectively. For $0 \leq i \leq k-2$ let $\gamma_i : G_i \longrightarrow G_{i+1}$ be a homomorphism and let $\gamma_{k-1} : G_{k-1} \longrightarrow G_0$ be a homomorphism. For all $n \in N$ let $G_n = G_{n \pmod{k}}$, $\gamma_n = \gamma_{n \pmod{k}}$ and $e_n = e_{n \pmod{k}}$.

2.3.3 For $m,t \in N$, with $t \geq 1$, let $\alpha_{m,t} = \gamma_m \gamma_{m+1} \cdots \gamma_{m+t-1}$ and let $\alpha_{m,o}$ be the identity automorphism on G_m. Thus for $m,t,s \in N$ we have

$$\alpha_{m,t}\, \alpha_{m+t,s} = \alpha_{m,s+t}$$

and $\alpha_{m,t} = \alpha_{m+sk,t}$

2.3.4 Fix $e \in E^*$. For each $f \in E^*$ define $v_f \in G_{\underline{f}+1}$ with v_{e+i} the identity of $G_{\underline{e+i}+1}$ for all $i \in N$. For all $t \in N$, with $t \geq 1$, and $f \in E^*$ define $m_{t,f} = (v_f\, \alpha_{\underline{f}+1,t-1})(v_{f+1}\, \alpha_{\underline{f}+2,t-2}) \cdots v_{f+t-1}$ and define $m_{o,f}$ to be the identity of $G_{\underline{f}}$. We note that $m_{t,f} \in G_{\underline{f}+t}$ for all $t \in N$, $f \in E^*$.

2.3.5 Let $S = \{(a,g_i,b) \in E^* \times (\bigcup_{i=0}^{k-1} G_i) \times E^*: \underline{a} = \underline{b} = i \text{ and } g_i \in G_i\}$ $\cup \{0\}$. Define a multiplication on S as follows:-

$$(a,g_i,b)(c,h_j,d) = (a+t,\ m_{t,a}^{-1}(g_i\alpha_{i,t})m_{t,b}m_{s,c}^{-1}(h_j\alpha_{j,s})m_{s,d},d+s)$$

where $t = [b,bc]$ and $s = [c,bc]$, if $bc \neq 0$; all other products are zero.

2.3.6 We show that this multiplication is closed. Let (a,g_i,b), $(c,h_j,d) \in S\setminus\{0\}$, If $bc = 0$, then $(a,g_i,b)(c,h_j,d) = 0 \in S$. We suppose, therefore, that $bc \neq 0$. Then

$$(a,g_i,b)(c,h_j,d) = (a+t,m_{t,a}^{-1}(g_i\alpha_{i,t})m_{t,b}m_{s,c}^{-1}(h_j\alpha_{j,s})m_{s,d},d+s)$$

where $t = [b,bc]$ and $s = [c,bc]$. Note that the outer elements in each triple (a,g_i,b), (c, h_j,d) are in S(E, T, k) and, when multiplied, behave exactly as there. Thus $\underline{a+t} = \underline{d+s}$. As $\underline{a} = \underline{b} = i$ we have $m_{t,a}$, $m_{t,b} \in G_{i+t}$ and similarly $m_{s,c}$, $m_{s,d} \in G_{j+s}$. However from Lemma 2.2.4 we have $\underline{a+t} \equiv \underline{a} + t \pmod k$ and $\underline{d+s} \equiv \underline{d} + s \pmod k$ so that $\underline{a+t} = \underline{d+s} \equiv i+t \pmod k \equiv j+s \pmod k$. Thus $m_{t,a},m_{t,b},m_{s,c}$, $m_{s,d} \in G_{\underline{a+t}}$. Furthermore $g_i\alpha_{i,t} \in G_{i+t}$ and $h_j\alpha_{j,s} \in G_{j+s}$ so that the middle term of the product $(a,g_i,b)(c,h_j,d)$ is a product of elements of $G_{\underline{a+t}}$ and so is in $G_{\underline{a+t}}$.

2.3.7 We denote the groupoid described in (2.3.5) and (2.3.6) by $S(E, T, k,G_i,\gamma_i,e,v_f)$.

2.4 $\underline{S(E, T, k, G_i,\gamma_i,e, v_f)}$

In this section we show that the groupoid described above is a 0-simple inverse semigroup with semilattice isomorphic to E and we examine the Green's relations on the semigroup.

2.4.1 <u>Theorem:</u> $S = S(E, T, k, G_i, \gamma_i, e, v_f)$ is a semigroup.

<u>Proof:</u> Having shown in (2.3.6) that the multiplication is closed we are left to consider the associativity. Let (a, g_i, b),

$(c,h_j,d),(f,1_n,g) \in S\setminus\{0\}$.

(a) If $bc = 0$ and $df = 0$, then $[(a,g_i,b)(c,h_j,d)](f,1_n,g) =$

$0\ (f,1_n,g) = 0$ and

$(a,g_i,b)[(c,h_j,d)(f,1_n,g)] = (a,g_i,b)\ 0 = 0$

(b) If $bc = 0$ and $df \neq 0$, then $[(a,g_i,b)(c,h_j,d)](f,1_n,g) =$

$0(f,1_n,g) = 0$.

On the other hand $(a,g_i,b)[(c,h_j,d)(f,1_n,g)] = (a,g_i,b)(c+t,x,g+s)$ where $t = [d,df], s = [f,df]$, and x is the appropriate middle term. However $b(c+t) = bc(c+t) = 0$ and so $(a,g_i,b)(c+t,x,g+s) = 0$.

(c) If $bc \neq 0$ and $df = 0$ we can show in a similar manner to (b) that $[(a,g_i,b)(c,h_j,d)](f,1_n,g) = (a,g_i,b)[(c,h_j,d)(f,1_n,g)] = 0$.

(d) The last case to consider is that with $bc \neq 0$ and $df \neq 0$. We examine first the product $[(a,g_i,b)(c,h_j,d)](f,1_n,g) =$

$(a+t, m_{t,a}^{-1}\ (g_i\alpha_{i,t})\ m_{t,b}\ m_{s,c}^{-1}\ (h_j\alpha_{j,s})\ m_{s,d}, d+s)(f,1_n,g)$ where

$t = [b,bc]$ and $s = [c,bc]$,

$= (a+t,x_p,d+s)(f,1_n,g)$, say, where $p = \underline{a+t}$,

$= ((a+t)+u,\ m_{u,a+t}^{-1}\ (x_p\alpha_{p,u})\ m_{u,d+s}\ m_{w,f}^{-1}\ (1_n\alpha_{n,w})\ m_{w,g}, g+w)$ where $u = [d+s,\ f(d+s)]$ and $w = [f,\ (d+s)f]$.

We now investigate $x_p\alpha_{p,u}$. We have, as $\alpha_{p,u}$ is a homomorphism,

$x_p\alpha_{p,u} = (m_{t,a}\ \alpha_{p,u})^{-1}\ (g_i\ \alpha_{i,t}\ \alpha_{p,u})\ (m_{t,b}\ \alpha_{p,u})(m_{s,c}\ \alpha_{p,u})^{-1}\ \times$

$(h_j\ \alpha_{j,s}\ \alpha_{p,u})(m_{s,d}\ \alpha_{p,u})$.

The following lemma simplifies this term considerably.

2.4.2 <u>Lemma</u>: If $a \in E^*$ and $t,s \in N$ then $m_{t,a}\alpha_{r,s} = m_{s+t,a}m_{s,a+t}^{-1}$ where $r = \underline{a+t}$.

<u>Proof</u>: By Lemma 2.2.4 we have $\underline{a+t} \equiv a+t \pmod k$. Thus from (2.3.3)

$$m_{t,a}\,\alpha_{r,s} = [(v_a\,\alpha_{\underline{a+1},t-1})(v_{a+1}\,\alpha_{\underline{a+2},t-2})\cdots(v_{a+t-1})]\;\alpha_{\underline{a+t},s}$$

$$= (v_a\,\alpha_{\underline{a+1},t-1}\,\alpha_{\underline{a+t},s})(v_{a+1}\,\alpha_{\underline{a+2},t-2}\,\alpha_{\underline{a+t},s})\cdots(v_{a+t-1}\alpha_{\underline{a+t},s})$$

$$= (v_a\,\alpha_{\underline{a+1},s+t-1})(v_{a+1}\,\alpha_{\underline{a+2},s+t-2})\cdots(v_{a+t-1}\,\alpha_{\underline{a+t},s}).$$

Hence we have

$$(m_{t,a}\,\alpha_{r,s})m_{s,a+t} = (v_a\,\alpha_{\underline{a+1},s+t-1})(v_{a+1}\,\alpha_{\underline{a+2},s+t-2})\ldots(v_{a+t-1}\alpha_{\underline{a+t},s})\times$$
$$(v_{a+t}\,\alpha_{\underline{a+t}+1,s-1})(v_{a+t+1}\,\alpha_{\underline{a+t}+2,s-2})\cdots v_{a+t+s-1}.$$
$$= (v_a\,\alpha_{\underline{a+1},s+t-1})(v_{a+1}\,\alpha_{\underline{a+2},s+t-2})\cdots\quad\times$$
$$(v_{a+t-1}\,\alpha_{\underline{a+t},s})(v_{a+t}\,\alpha_{\underline{a+t+1},s-1})\times$$
$$(v_{a+t+1}\,\alpha_{\underline{a+t+2},s-2})\cdots\,v_{a+t+s-1}$$ as, by

Lemma 2.2.4, $\underline{a+t} \equiv \underline{a}+t \pmod k$.

Thus $(m_{t,a}\,\alpha_{r,s})m_{s,a+t} = m_{s+t,a}$. From this we have $(m_{t,a}\,\alpha_{r,s})m_{s,a+t}\,m^{-1}_{s,a+t} = m_{s+t,a}\,m^{-1}_{s,a+t}$. However $m_{s,a+t} \in G_{\underline{a+t+s}}$ and $m_{t,a}\,\alpha_{r,s} \in G_{\underline{a+t+s}}$ so that $(m_{t,a}\,\alpha_{r,s})m_{s,a+t}m^{-1}_{s,a+t} = m_{t,a}\alpha_{r,s}$

Applying this lemma four times we can simplify $x_p\,\alpha_{p,u}$.

(1) $m_{t,a}\,\alpha_{p,u} = m_{t+u,a}\,m^{-1}_{u,a+t}$

(2) We have $p = \underline{a+t} \equiv \underline{a}+t \pmod k \equiv \underline{b}+t \pmod k \equiv \underline{b+t} \pmod k$ by Lemma 2.2.4. Thus $m_{t,b}\;\alpha_{p,u} = m_{t+u,b}\,m^{-1}_{u,b+t} = m_{t+u,b}\,m^{-1}_{u,bc}$.

(3) We have $p = \underline{a+t} = \underline{d+s} \equiv \underline{d}+s \pmod k \equiv \underline{c}+s \pmod k \equiv \underline{c+s} \pmod k$ by Lemma 2.2.4. Thus $m_{s,c}\,\alpha_{p,u} = m_{s+u,c}\,m^{-1}_{u,c+s} = m_{s+u,c}\,m^{-1}_{u,bc}$

(4) $m_{s,d}\,\alpha_{p,u} = m_{s+u,d}\,m^{-1}_{u,d+s}$.

We note further that $\alpha_{i,t}\,\alpha_{p,u} = \alpha_{i,t}\,\alpha_{i+t,u} = \alpha_{i,t+u}$ since $p = \underline{a+t} \equiv \underline{a}+t \pmod k$ by Lemma 2.2.4 and using (2.3.3). Also, by a similar argument, $\alpha_{j,s}\,\alpha_{p,u} = \alpha_{j,s}\,\alpha_{j+s,u} = \alpha_{j,s+u}$.

Combining all these results we have

$$x_p \alpha_{p,u} = m_{u,a+t}\, m^{-1}_{t+u,a}\, (g_i\, \alpha_{i,t+u}) m_{t+u,b}\, m^{-1}_{u,bc} m_{u,bc}\, m^{-1}_{s+u,c}\ \chi$$
$$(h_j\, \alpha_{j,s+u})\, m_{s+u,d}\, m^{-1}_{u,d+s}\ .$$

The middle term of the product $[(a,g_i,b)(c,h_j,d)](f,1_n,g)$ is thus

$$m^{-1}_{t+u,a}\, (g_i\, \alpha_{i,t+u})\, m_{t+u,b}\, m^{-1}_{s+u,c} (h_j\, \alpha_{j,s+u}) m_{s+u,d}\, m^{-1}_{w,f}\, (1_n \alpha_{n,w}) m_{w,g}$$

If we now examine the product $(a,g_i,b)[(c,h_j,d)(f,1_n,g)]$ we find by similar arguments that it is

$$(a+y,\ m^{-1}_{y,a} (g_i \alpha_{i,y}) m_{y,b}\, m^{-1}_{z+r,c} (h_j \alpha_{j,z+r}) m_{z+r,d}\, m^{-1}_{x+z,f} (1_n \alpha_{n,x+z})\ \chi$$
$$m_{x+z,g},\ (g+x)+z)$$

where $r = [d,df]$, $x = [f,df]$, $y = [b,b(c+r)]$ and $z = [c+r,b(c+r)]$.

As we have earlier noted, the outer components of each term of $S(E, T, k, G_i, \gamma_i, e, v_f)$ are in $S(E, T, k)$ and, under multiplication, behave exactly as there. Since multiplication is associative in $S(E, T, k)$ we thus have $a + t + u = a+y$ and $g+w = g+x+z$ so that $t+u = y$ and $w = x+z$. To complete the proof that the middle components of the two products are equal we need now only show that $s+u = z+r$. We note that $z+r = [c+r,\ b(c+r)] + [c,c+r] = [c,\ b(c+r)] = [c,bc] + [bc,\ b(c+r)] = s + [b,\ b(c+r)] - [b,bc] = s + y-t = s + (t+u)-t = s+u$.

<u>2.4.3 Lemma</u> In $S = S(E, T, k, G_i, \gamma_i, e, v_f)$ the element (a,g_i,b) is an idempotent if and only if $a = b$ and $g_i = e_i$.

<u>Proof:</u> Let $(a,g_i,b) \in S$ and be an idempotent. Then $(a,g_i,b) = (a,g_i,b)(a,g_i,b)$. Since $(a,g_i,b) \neq 0$ we must have $ab \neq 0$ and so $(a,g_i,b) = (a+t,\ m^{-1}_{t,a} (g_i\ \alpha_{i,t})\ m_{t,b}\ m^{-1}_{s,a} (g_i \alpha_{i,s})\ m_{s,b}, b+s)$ where $t = [b,ab]$ and $s = [a,ab]$. However $a = a+t$ and $b = b+s$ so that $s = t = 0$ and we have $b = ab = a$. Also $(a,g_i,b) = (a,g_i g_i,b)$ so that $g_i = g_i^2$ and $g_i = e_i$.

Conversely let $(a,e_i,a) \in S$. Then $(a,e_i,a)(a,e_i,a) = (a,e_ie_i,a) = (a,e_i,a)$.

2.4.4 <u>Theorem:</u> The semigroup $S = S(E, T, k, G_i, \gamma_i, e, v_f)$ is a 0-simple inverse semigroup.

<u>Proof:</u> Let $(a,g_i,b) \in S \setminus \{0\}$ and consider $(a,g_i,b)(b,g_i^{-1},a)(a,g_i,b) = (a,e_i,a)(a,g_i,b) = (a,g_i,b)$. Thus S is regular. To complete the proof that S is inverse we need only by (1.1.4) check that the idempotents in S commute. Let (a,e_i,a) and (c,e_j,c) be two idempotents in S. Then $(a,e_i,a)(c,e_j,c) = 0 = (c,e_j,c)(a,e_i,a)$ if $ac = 0$. If $ac \neq 0$, $(a,e_i,a)(c,e_j,c) = (a+t, m_{t,a}^{-1}(e_i\alpha_{i,t})\, m_{t,a}\, m_{s,c}^{-1} \times (e_j\alpha_{j,s})\, m_{s,c}, c+s)$ where $t = [a,ac]$ and $s = [c,ac]$. We have $e_i\alpha_{i,t} = e_{i+t}$ and $e_j\alpha_{j,s} = e_{j+s}$. However $i+t = \underline{a}+t \equiv \underline{a+t} \pmod k$, by Lemma 2.2.4, and $j+s = \underline{c}+s \equiv \underline{c+s} \pmod k$ so that $i+t \equiv j+s \pmod k$ and so $e_i\alpha_{i,t} = e_j\alpha_{j,s}$. Thus $(a,e_i,a)(c,e_j,c) = (ac,e_{i+t},ac)$. We can show similarly that $(c,e_j,c)(a,e_i,a) = (ca,e_{i+t},ca) = (ac,e_{i+t},ac)$.

Let (a,g_i,b), $(c,h_j,d) \in S \setminus \{0\}$. Then (a,b), $(c,d) \in S(E, T, k)$ which is, by Theorem 2.2.3, a 0-subtransitive inverse subsemigroup of T_E. Since E is an ω-tree with zero it is 0-subuniform and so we can apply [9, Theorem 3.2 (ii)] and we have that S(E, T, k) is 0-simple. Thus there exist (w,x), $(y,z) \in S(E, T, k)$ such that $(a,b) = (w,x)(c,d) \times (y,z)$. Also $(w,e_{\underline{w}},x)(c,h_j,d)(y,e_{\underline{z}},z) = (a,p_i,b)$, say, where p_i is the appropriate middle term. Hence $(a,g_i,b) = (w,e_{\underline{w}},x)(c,h_j,d)(y,e_{\underline{z}},z) \times (b,p_i^{-1}g_i,b)$ and so we have S is 0-simple.

In the following theorem we examine in detail the semigroup $S(E, T, k, G_i, \gamma_i, e, v_f)$.

2.4.5 <u>Theorem</u>: In $S = S(E, T, k, G_i, \gamma_i, e, v_f)$

1. $E_S \cong E$

2. $((a,g_i,b),(c,h_j,d)) \in \mathcal{R} \iff a = c$

   $((a,g_i,b),(c,h_j,d)) \in \mathcal{L} \iff b = d$

   $((a,g_i,b),(c,h_j,d)) \in \mathcal{H} \iff a = c$ and $b = d$

   $((a,g_i,b),(c,h_j,d)) \in \mathcal{D} \iff i = j$

3. $\mathcal{H}$ is a congruence on S

4. The maximal subgroups of S are, to within isomorphism, $G_0, G_1, G_2, \ldots, G_{k-1}$.

5. There are exactly k non zero $\mathcal{D}$-classes in S.

<u>Proof</u>: 1. From Lemma 2.4.3 we have $E_S = \{(a,e_i,a) \in S \setminus \{0\}\} \cup \{0\}$.
Define a mapping $\phi: E \to E_S$ as follows:-
$0\phi = 0$ and $a\phi = (a,e_{\underline{a}},a)$.
Clearly ϕ is a bijection. If $a,b \in E^*$ with $ab = 0$ then $(ab)\phi = 0$ and $(a\phi)(b\phi) = 0$. If $a,b \in E^*$ with $ab \neq 0$ then $(a\phi)(b\phi) = (a,e_{\underline{a}},a)(b,e_{\underline{b}},b) = (ab,e_{\underline{a}+t},ab)$ where $t = [a,ab]$, from the proof of Theorem 2.4.4. Thus $(a\phi)(b\phi) = (ab)\phi$ and we have ϕ is a homomorphism and hence an isomorphism.

2. Let $(a,g_i,b),(c,h_j,d) \in S$ with $((a,g_i,b),(c,h_j,d)) \in \mathcal{R}$.
We have $((a,g_i,b),(a,g_i,b)(a,g_i,b)^{-1}) \in \mathcal{R}$, i.e. $((a,g_i,b),(a,e_i,a)) \in \mathcal{R}$.
Similarly $((c,h_j,d),(c,e_j,c)) \in \mathcal{R}$. Thus $((a,e_i,a),(c,e_j,c)) \in \mathcal{R}$.
However from (1.2.7) we now have $(a,e_i,a) = (c,e_j,c)$ so that $a = c$.

Conversely let $(a,g_i,b),(a,h_i,d) \in S$. Then, as above, $((a,g_i,b),(a,e_i,a)) \in \mathcal{R}$ and $((a,h_i,d),(a,e_i,a)) \in \mathcal{R}$ so that $((a,g_i,b),(a,h_i,d)) \in \mathcal{R}$.

The result for $\mathcal{L}$ follows similarly and the result for $\mathcal{H}$ can then be readily deduced.

Let $(a,g_i,b),(c,h_j,d) \in S$ with $((a,g_i,b),(c,h_j,d)) \in \mathcal{D}$. Then there exists $(f,l_t,g) \in S$ such that $((a,g_i,b),(f,l_t,g)) \in \mathcal{R}$ and $((f,l_t,g),(c,h_j,d)) \in \mathcal{L}$. From the above results we have $a = f$ and $g = d$. Hence $\underline{a} = \underline{f}$ and $\underline{g} = \underline{d}$. However $\underline{f} = \underline{g}$ and so we have $\underline{a} = \underline{d}$, i.e. $i = j$.

Conversely let $(a,g_i,b),(c,h_i,d) \in S$. From the results on $\mathcal{R}$ and $\mathcal{L}$ above we have $((a,g_i,b),(a,e_i,d)) \in \mathcal{R}$ and $((a,e_i,d)(c,h_i,d)) \in \mathcal{L}$ so that $((a,g_i,b),(c,h_i,d)) \in \mathcal{D}$.

3. This result can be checked easily.

4. From (1.2.11) we have that the maximal subgroups of S are the $\mathcal{H}$-classes of S containing idempotents. From result 2 above, $H_{(a,e_i,a)} = \{(a,g_i,a):g_i \in G_i\} \cong G_i$ and so the maximal subgroups of S are isomorphic to the groups $G_0,G_1,\ldots,G_{k-1}$.

5. For $i = 0,1,2,\ldots,k-1$ let $D_i = \{(a,g_i,b):\underline{a} = \underline{b} = i$ and $g_i \in G_i\}$. From result 2 above, for any $i \in N$, where $0 \leq i \leq k-1$, any two elements in D_i are $\mathcal{D}$-equivalent. Further if $(a,g_i,b),(p,x_r,q) \in S$ and $((a,g_i,b),(p,x_r,q)) \in \mathcal{D}$ then $\underline{p} = \underline{q} = r = i$ and $(p,x_r,q) \in D_i$. Thus the non zero $\mathcal{D}$-classes of S are precisely the sets $D_0,D_1,\ldots,D_{k-1}$ and so there are exactly k non zero $\mathcal{D}$-classes of S.

2.4.6 <u>Corollary</u>: In S(E, T, k) there are exactly k non zero $\mathcal{D}$-classes.

<u>Proof:</u> It is sufficient to say that $S(E, T, k) = S(E, T, k, G_i,\gamma_i, e, v_f)$ where, for $0 \leq i \leq k-1$, $G_i = \{e_i\}$.

2.5 A 0-simple inverse semigroup whose semilattice is an ω-tree with zero.

We have shown in section 2.4 that the construction

$S(E, T, k, G_i, \gamma_i, e, v_f)$ gives rise to a 0-simple inverse semigroup whose semilattice is an ω-tree with zero. In this section we prove that, in fact, every 0-simple inverse semigroup whose semilattice is an ω-tree with zero is of the form described above. The result is reached in two stages: the first stage is the consideration of the fundamental semigroup $S/\mathcal{H}$ and the second stage is the consideration of S itself.

2.5.1 Theorem: Let S be a 0-simple inverse semigroup whose semilattice E is an ω-tree with zero. Then $\mathcal{H}$ is a congruence on S and there exists a transversal T of the components of E^* and $k \in N$, with $k \geq 1$, such that $S/\mathcal{H} \cong S(E, T, k)$.

Proof: From [6, Theorem 3.2] we have that $\mathcal{H} = \mu$ and so $\mathcal{H}$ is a congruence on S. As E is an ω-tree with zero, E satisfies the conditions of (1.3.11) and we have that $T_E = \{\xi_{e,f}: e,f \in E^*\} \cup \{0\}$ as described in (1.3.10) and (1.3.11). Applying [6, Lemma 3.1] we have $S/\mathcal{H} = S/\mu \cong S\theta$ where $\theta: S \to T_E$ is a homomorphism with $0\theta = 0$ and $a\theta = \xi_{aa^{-1},a^{-1}a}$ for $a \in S \setminus \{0\}$. If, as in (1.3.11), we take $T_E = \{(e,f) : e,f \in E^*\} \cup \{0\}$ then $S\theta$ is the set $\{(aa^{-1},a^{-1}a) : a \in S \setminus \{0\}\} \cup \{0\}$. We note that, by [9, Theorem 2.4], $S/\mathcal{H}$ is a fundamental inverse semigroup with semilattice isomorphic to E. Hence $S\theta$ is a fundamental 0-simple inverse subsemigroup of T_E. If we now apply [9, Theorem 3.2 (i)] we have that $S\theta$ is a 0-subtransitive inverse subsemigroup of T_E. Then, by Theorem 2.2.5, there exist $k \in N$, with $k \geq 1$, and a transversal T of the components of E^* such that $S\theta = S(E, T, k)$.

2.5.2 With the same notation as in 2.2 let the isomorphism discussed in Theorem 2.5.1 be $\phi : S/\mathcal{H} \to \{(f,g) : f,g \in E^* \text{ and } \underline{f} = \underline{g}\} \cup \{0\}$ where $0\phi = 0$ and $(H_a)\phi = (aa^{-1},a^{-1}a)$.

2.5.3 <u>Theorem</u>: Let S be a 0-simple inverse semigroup whose semilattice E is an ω-tree with zero. Then there exists a semigroup $S(E, T, k, G_i, \gamma_i, e, v_f)$ such that $S \cong S(E, T, k, G_i, \gamma_i, e, v_f)$.

<u>Proof</u>: It has been shown in Theorem 2.5.1 that there exist $k \in N$, with $k \geq 1$, and T a transversal of the components of E^* such that $\phi : S/\mathcal{H} \to S(E, T, k)$, described in (2.5.2), is an isomorphism. With k and T as there, select an element $e \in E^*$ for which $\underline{e} = 0$ and keep it fixed.

For $i = 0,1,2,\ldots,k-1$ let $G_i = H_{e+i}$ and, for all $n \in N$, take $G_n = G_{n \pmod k}$. Then, for all $n \in N$, G_n is a group.

We next choose a set of representatives of the non-zero $\mathcal{H}$-classes of S as follows:-

For $f \in E^*$, with $\underline{f} = i$, let u_f be the representative of H_x where $(H_x)\phi = (e+i,f)$. We make the following stipulations:-

(a) u_{e+i} is the identity of G_i for $i \in N$ with $0 \leq i \leq k-1$.

(b) $u_{e+nk} = u^n_{e+k} = u^n$ (say) for all $n \in N$, with $n \geq 1$.

(c) $u_{e+m+nk} = (e+m)u^n$ for all $m,n \in N$ with $n \geq 1$ and $0 \leq m \leq k-1$.

(d) $u_f^{-1}u_g$ is the representative of H_y where $(H_y)\phi = (f,g)$.

We note that if $f \in E^*$ with $\underline{f} = i$ then $u_f u_f^{-1} = e+i$ and $u_f^{-1}u_f = f$.

In the next lemma we obtain a method of expressing all the elements of $S \setminus \{0\}$ in terms of these representatives and elements of the groups G_i, $i = 0,1,2,\ldots,k-1$.

2.5.4 <u>Lemma</u>: Let $x \in S \setminus \{0\}$. Then there exists a unique representation of x in the form $u_f^{-1}g_i u_h$ where $(H_x)\phi = (f,h)$, $\underline{f} = \underline{h} = i$ and $g_i \in G_i$.

<u>Proof</u>: Let $x \in S\setminus \{0\}$ with $(H_x)\phi = (f,h)$. Since $(f,h) \in S(E, T, k)$, $\underline{f} = \underline{h} = i$ (say). We thus have $(f, e+i)(e+i,e+i)(e+i,h) = (f,h)$ in

$S(E, T, k)$ and so $(H_x)\phi = (H_{u_f^{-1}} H_{e+i} H_{u_h})\phi$. Since $\mathcal{H}$ is a congruence on S, by Theorem 2.5.1, we have $(H_x)\phi = (H_{u_f^{-1}(e+i)u_h})\phi$. However ϕ is an isomorphism and so $H_x = H_{u_f^{-1}(e+i)\,u_h}$. Thus $(x,\ u_f^{-1}(e+i)u_h) \in \mathcal{H}$. Again, using the fact that $\mathcal{H}$ is a congruence, we have $(u_f x u_h^{-1},\ u_f u_f^{-1}(e+i)u_h u_h^{-1}) \in \mathcal{H}$, so that, as $u_f u_f^{-1} = e+i = u_h u_h^{-1}$, we have $(u_f\, x\, u_h^{-1}, e+i) \in \mathcal{H}$. However $G_i = H_{e+i}$ and so $u_f\, x\, u_h^{-1} = g_i$ (say), where $g_i \in G_i$. Thus $u_f^{-1}u_f\, x\, u_h^{-1}u_h = u_f^{-1}g_i u_h$. However $u_f^{-1}u_f = f$ and $u_h^{-1}u_h = h$, also $xx^{-1} = f$ and $x^{-1}x = h$ and we have $fxh = x$. Thus $x = u_f^{-1}\ g_i u_h$. Hence there is a representation of x in the required form.

Suppose that $x \in S \setminus \{0\}$ and x has two representations in the required form, the first being $x = u_f^{-1}\ g_i u_h$, and the second being $x = u_p^{-1}h_j u_q$. From $x = u_f^{-1}g_i u_h$ we have $(H_x)\phi = (f,h)$ and from $x = u_p^{-1}h_j u_q$ we have $(H_x)\phi = (p,q)$. Thus $(f,h) = (p,q)$ and $f = p$ and $h = q$. Consequently $i = \underline{f} = \underline{p} = j$ and we now have $u_f^{-1}\ g_i u_h = x = u_f^{-1}h_i u_h$. However, $u_f\ x\ u_h^{-1} = (e+i)g_i(e+i) = g_i$ and $u_f\ x\ u_h^{-1} = (e+i)h_i(e+i) = h_i$ so that $g_i = h_i$ and the representation is unique.

Returning to the theorem we now use this representation to define a mapping ψ: $S \to \{(f,g_i,h) : f,h \in E^*,\ \underline{f} = \underline{h} = i$ and $g_i \in G_i\} \cup \{0\}$ as follows:- $0\psi = 0$

$x\psi = (f,g_i,h)$ where $(H_x)\phi = (f,h)$ and the representation of x described in Lemma 2.5.4 is $u_f^{-1}g_i u_h$. From Lemma 2.5.4 we see that the mapping ψ is well-defined. It is also readily seen to be a surjection, for, if $(p,g_i,q) \in E^* \times \left(\bigcup_{i=0}^{k-1} G_i\right) \times E^*$, with $\underline{p} = \underline{q} = i$ and $g_i \in G_i$ then, letting $y = u_p^{-1}\ g_i u_q$ we have $y\psi = (p,g_i,q)$. Also if $x,y \in S \setminus \{0\}$ and $x\psi = y\psi$ we have at once that $x = y$.

Let $i \in N$ with $0 \le i \le k-2$ and let $g_i \in G_i$. Then $(H_{g_i(e+i+1)})\phi = (e+i,e+i)(e+i+1,e+i+1) = (e+i+1,e+i+1)$. Thus $g_i(e+i+1) = (e+i+1)g_i(e+i+1)$. Similarly $(e+i+1)g_i = (e+i+1)g_i \times (e+i+1)$. Hence $(e+i+1)g_i = g_i(e+i+1)$ and is in G_{i+1}. We define a mapping $\gamma_i : G_i \to G_{i+1}$ by the rule that $g_i\gamma_i = g_i(e+i+1)$. It is immediate from the remarks above that γ_i is a well-defined homomorphism.

For $g_{k-1} \in G_{k-1}$ we examine $(H_{ug_{k-1}})\phi = (e,e+k)(e+k-1, e+k-1) = (e,e+k)$. By Lemma 2.5.4 we thus have $ug_{k-1} = u_e^{-1}g_o u$ for some unique $g_o \in G_o$. Thus $ug_{k-1} = g_o u$. Define a mapping $\gamma_{k-1} : G_{k-1} \to G_o$ by the rule that $(g_{k-1}\gamma_{k-1})u = ug_{k-1}$. This is again easily seen to be a well-defined homomorphism.

We now extend these definitions taking $\gamma_n = \gamma_{n(\text{mod } k)}$ for all $n \in N$. If $n,t \in N$ with $t \ge 1$ define $\alpha_{n,t} = \gamma_n\gamma_{n+1}\cdots\gamma_{n+t-1}$ and $\alpha_{n,o}$ to be the identity automorphism on G_n. Note that for $m, s, t \in N$ we have

$$\left.\begin{aligned} \alpha_{m,t}\,\alpha_{m+t,s} &= \alpha_{m,t+s} \\ \text{and } \alpha_{m,t} &= \alpha_{m+sk,t} \end{aligned}\right\} \quad 2.5.3\ (i)$$

If $0 \le i \le j \le k-1$ we have $g_i(e+j) = g_i(e+i+1)(e+i+2)\ldots(e+j)$ so that $g_i(e+j) = g_i\gamma_i\gamma_{i+1}\cdots\gamma_{j-1} = g_i\alpha_{i,j-i}$ ... 2.5.3 (ii).
Similarly we have $(e+j)g_i = g_i\alpha_{i,j-i}$ ...2.5.3 (iii).

The next lemma is concerned with these homomorphisms.

2.5.5 <u>Lemma</u>: If $n,i \in N$ with $n \ge 1$, $0 \le i \le k-1$ and $g_i \in G_i$ then $u^n g_i = (g_i\alpha_{i,nk-i})u^n$.

<u>Proof</u>: We commence an inductive proof by considering the case when $n = 1$. We then have $(H_{ug_i})\phi = (e, e+k)(e+i,e+i) = (e,e+k)$ and

so $ug_i = (ug_i)(e+k) = (ug_i)(e+k)(e+k-1) = (ug_i)(e+k-1) = u(g_i(e+k-1))$. However, by (2.5.3 (ii)), $g_i(e+k-1) = g_i\alpha_{i,k-1-i}$ and $g_i(e+k-1) \in G_{k-1}$. Hence $ug_i = u(g_i\alpha_{i,k-1-i}) = ((g_i\alpha_{i,k-1-i})\gamma_{k-1})u = (g_i\alpha_{i,k-i})u$. The proposition is therefore true in the case $n = 1$.

Assume that the proposition is true for $n = r-1$ where $r \geq 2$ and consider the case $n = r$. In the case $n = r-1$ we have the result that, if $0 \leq i \leq k-1$ and $g_i \in G_i$, then $u^{r-1}g_i = (g_i\alpha_{i,(r-1)k-i})u^{r-1}$. When $n = r$, we consider $u^r g_i = u^{r-1}(ug_i) = u^{r-1}(g_i\alpha_{i,k-i})u$, since the proposition is true for $n = 1$. Applying the proposition for $n = r-1$ we now have $u^r g_i = ((g_i\alpha_{i,k-i})\alpha_{o,(r-1)k}\ u^{r-1})u$ since $g_i\alpha_{i,k-i} \in G_o$. Hence, by (2.5.3 (i)), we have $u^r g_i = (g_i\alpha_{i,rk-i})u^r$ and we have proved the proposition for $n = r$. Thus, by induction, for all $n \in N$ with $n \geq 1$, the proposition holds.

Returning once more to the theorem we now make a notational definition:-

Let $f \in E^*$ and $t \in N$. If $i = \underline{f}$ and $p = \underline{f+t}$, we define $m_{t,f}$ to be the unique element in G_p such that $u_{e+i+t}u_f = m_{t,f}u_{f+t}$. This is a valid definition, since $(H_{u_{e+i+t}u_f})\phi = (e+p,e+i+t)\chi$ $(e+i,f) = (e+p,f+t)$ and so, by Lemma 2.5.4, $u_{e+i+t}u_f = u^{-1}_{e+p}g_p u_{f+t}$ for some unique element $g_p \in G_p$. However $u_{e+p} = e+p$ and so we have $u_{e+i+t}u_f = g_p u_{f+t}$. We take $m_{t,f} = g_p$.

For all $f \in E^*$ we denote by v_f the element $m_{1,f}$. We note that $m_{o,f}u_f = u_{e+i}u_f = u_f$ so that $m_{o,f} = e+i$, where $\underline{f} = i$.

We now show that this notational definition is a suitable one for the construction of the semigroup $S(E, T, k, G_i, \gamma_i, e, v_f)$.

2.5.6 <u>Lemma</u>: Let $f,h \in E^*$ with $\underline{f} = i$ and $\underline{h} = j$. Then, if $fh \neq 0$, $u_f u_h^{-1} = u^{-1}_{e+i+t}\ m_{t,f}\ m^{-1}_{s,h}\ u_{e+j+s}$ where $t = [f,fh]$ and $s = [h,fh]$ and, if $fh = 0$, $u_f u_h^{-1} = 0$.

<u>Proof</u>: If $fh \neq 0$ then $(H_{u_f u_h^{-1}})\phi = (e+i,f)(h,e+j) = (e+i+t, e+j+s)$ where $t = [f,fh]$ and $s = [h,fh]$. Since this product is in

$S(E, T, k)$ we have $\underline{e+i+t} = \underline{e+j+s} = p$ (say). By Lemma 2.5.4, $u_f u_h^{-1} = u_{e+i+t}^{-1}\, g_p\, u_{e+j+s}$ where $g_p \in G_p$. From this we see that $u_{e+i+t}\, u_f\, u_h^{-1}\, u_{e+j+s}^{-1} = (e+p)g_p(e+p) = g_p$. However, $u_{e+i+t}\, u_f = m_{t,f}\, u_{f+t}$ and $u_{e+j+s}\, u_h = m_{s,h}\, u_{h+s}$. Thus $g_p = m_{t,f}\, u_{f+t}\, u_{h+s}^{-1} \times m_{s,h}^{-1}$. Noting that $f+t = h+s = fh$, we have $g_p = m_{t,f}\,(e+p)\, m_{s,h}^{-1} = m_{t,f}\, m_{s,h}^{-1}$, as $m_{t,f} \in G_{\underline{f+t}} = G_p$ since $\underline{f+t} \equiv \underline{f} + t \pmod{k} \equiv i+t \pmod{k} \equiv \underline{e+i+t} \pmod{k}$, by Lemma 2.2.4. From this we have $u_f\, u_h^{-1} = u_{e+i+t}^{-1}\, m_{t,f}\, m_{s,h}^{-1}\, u_{e+j+s}$.

If $fh = 0$ then $(H_{u_f u_h^{-1}})\phi = (e+i,f)(h,e+j) = 0$ and so $u_f u_h^{-1} = 0$.

2.5.7 <u>Lemma</u>: Let $f \in E^*$ with $\underline{f} = i$ and let $t \in N$ with $t \geq 1$. Then $m_{t,f} = (v_f \alpha_{i+1,t-1})(v_{f+1}\alpha_{i+2,t-2}) \cdots\cdots (v_{f+t-1})$.

<u>Proof</u>: The proposition holds for all $f \in E^*$ when $t = 1$. Assume that the proposition holds for $t \in N$, with $t \geq 1$, and $f \in E^*$ where $\underline{f} = i$, i.e. that $m_{t,f} = (v_f\alpha_{i+1,t-1})(v_{f+1}\alpha_{i+2,t-2}) \cdots (v_{f+t-1})$. By definition, $m_{t+1,f}$ is such that $m_{t+1,f}\, u_{f+t+1} = u_{e+i+t+1}\, u_f$, where $\underline{f+t+1} = q$ (say). However $m_{t,f}\, u_{f+t} = u_{e+i+t}\, u_f$, where $\underline{f+t} = p$ (say). By Lemma 2.2.4, $q \equiv \underline{f+t} + 1 \pmod{k} \equiv p+1 \pmod{k}$. There are therefore two cases to consider:-

(i) $0 \leq p \leq k-2$ and $q = p+1$

(ii) $p = k-1$ and $q = 0$

Case (i) Let $i+t = rk+p$ where $r \in N$. Then $m_{t+1,f}\, u_{f+t+1} = u_{e+rk+p+1}\, u_f = (e+p+1)u^r u_f = (e+p+1)(e+p)u^r u_f = (e+p+1) \times u_{e+rk+p}\, u_f = (e+p+1)\, m_{t,f}\, u_{f+t}$. Since $m_{t,f} \in G_p$ it follows that $m_{t+1,f}\, u_{f+t+1} = (m_{t,f})\gamma_p\, u_{f+t}$. Hence $m_{t+1,f}\, u_{f+t+1}\, u_{f+t+1}^{-1} = (m_{t,f})\gamma_p\, u_{f+t}\, u_{f+t+1}^{-1}$. However $u_{f+t+1}\, u_{f+t+1}^{-1} = (e+p+1)$ and so $m_{t+1,f} = (m_{t,f})\gamma_p\, u_{f+t}\, u_{f+t+1}^{-1}$. By Lemma 2.5.6 we have

$u_{f+t}\, u^{-1}_{f+t+1} = u^{-1}_{e+p+1}\, m_{1,f+t}\, m^{-1}_{0,f+t+1}\, u_{e+p+1} = (e+p+1)\, m_{1,f+t} \times$ $(e+p+1)(e+p+1) = m_{1,f+t} = v_{f+t}$. Thus $m_{t+1,f} = (m_{t,f})\, \gamma_p\, v_{f+t}$ which is the required result in this case.

Case (ii) Let $i+t = rk+k-1$ where $r \in N$. Then $m_{t+1,f}\, u_{f+t+1} = u_{e+(r+1)k}\, u_f = u^{r+1} u_f = uu^r u_f = u(e+k-1)u^r u_f = uu_{e+rk+k-1} u_f = u(m_{t,f}\, u_{f+t}) = (m_{t,f})\, \gamma_{k-1}\, uu_{f+t}$, since $m_{t,f} \in G_{k-1}$. Thus we have $m_{t+1,f}\, u_{f+t+1}\, u^{-1}_{f+t+1} = (m_{t,f})\, \gamma_{k-1}\, uu_{f+t} u^{-1}_{f+t+1}$. However $u_{f+t+1}\, u^{-1}_{f+t+1} = e$ and we have $m_{t+1,f} = (m_{t,f})\gamma_{k-1}\, uu_{f+t} u^{-1}_{f+t+1}$. By Lemma 2.5.6 we have $u_{f+t}\, u^{-1}_{f+t+1} = u^{-1}_{e+k}\, m_{1,f+t}\, m^{-1}_{o,f+t+1}\, u_e = u^{-1}\, m_{1,f+t}$. Thus $m_{t+1,f} = (m_{t,f})\, \gamma_{k-1}\, uu^{-1}\, m_{1,f+t} = (m_{t,f})\gamma_{k-1} \times$ $m_{1,f+t} = (m_{t,f})\, \gamma_{k-1}\, v_{f+t}$ which is the required result.

We return now to the theorem. The final step in showing that the v_f's defined above satisfy the requirements for the semigroup $S(E, T, k, G_i, \gamma_i, e, v_f)$ is to show that, if $i \in N$, v_{e+i} is the identity of the group G_{i+1}. Let $i \in N$ with $i = rk + p$ where $r \in N$ and $0 \leq p \leq k-1$. Then $v_{e+i} = m_{1,e+i}$ and so $v_{e+i}\, u_{e+i+1} = u_{e+p+1}\, u_{e+i}$, where $q = \underline{e+i+1}$ (say). Thus $v_{e+i} = u_{e+p+1}\, u_{e+i}\, u^{-1}_{e+i+1}$. By Lemma 2.2.4, we have $q \equiv \underline{e+i}+1 \pmod k$ $\equiv p + 1 \pmod k$. Hence there are two cases to be considered here:-

(i) that $0 \leq p \leq k-2$ and $q = p+1$

(ii) that $p = k-1$ and $q = 0$.

Case (i). We have $v_{e+i} = (e+p+1)\, u_{e+rk+p}\, u^{-1}_{e+i+1} = (e+p+1) \times$ $(e+p)\, u^r\, u^{-1}_{e+i+1} = (e+p+1)\, u^r\, u^{-1}_{e+i+1} = u_{e+rk+p+1}\, u^{-1}_{e+i+1} = u_{e+i+1}\, u^{-1}_{e+i+1} = (e+p+1)$.

Case (ii). We have $v_{e+i} = u_{e+k}\, u_{e+i}\, u^{-1}_{e+i+1} = u\,(e+k-1)u^r\, u^{-1}_{e+i+1} = u(e+k)(e+k-1)u^r\, u^{-1}_{e+i+1} = u(e+k)u^r\, u^{-1}_{e+i+1} = u^{r+1}\, u^{-1}_{e+i+1} = u_{e+(r+1)k}\, u^{-1}_{e+(r+1)k} = e$. In both cases we have the required result.

It follows now that, with T, k, G_i, γ_i, e, v_f as specified above, the semigroup S(E, T, k, G_i, γ_i, e, v_f) can be defined. The mapping ψ described above is clearly a bijection from S onto S(E, T, k, G_i, γ_i, e, v_f) and the remainder of the proof is concerned with showing that ψ is an isomorphism.

Let $x,y \in S\setminus\{0\}$ with $x\psi = (a,g_i,b)$ and $y\psi = (c,h_j,d)$. Then $x = u_a^{-1} g_i u_b$ and $y = u_c^{-1} h_j u_d$.

If bc = 0 then $(x\psi)(y\psi) = 0$. Also $(H_x)\phi(H_y)\phi = 0$ and so $(H_{xy})\phi = 0$ from which we have xy = 0. Hence $(xy)\psi = (x\psi)(y\psi)$.

If bc ≠ 0 then $(H_{xy})\phi = (H_x)\phi(H_y)\phi = (a,b)(c,d) = (a+t,d+s)$ where t = [b,bc] and s = [c,bc]. Thus $(xy)\psi = (a+t, z, d+s)$ where $xy = u_{a+t}^{-1} z u_{d+s}$ with <u>a+t</u> = <u>d+s</u> = p (say) and $z \in G_p$. From this we have $u_{a+t}\, xy\, u_{d+s}^{-1} = u_{a+t} u_{a+t}^{-1} z u_{d+s} u_{d+s}^{-1}$ (e+p) z (e+p) = z. Hence $z = u_{a+t} u_a^{-1} g_i u_b u_c^{-1} h_j u_d u_{d+s}^{-1}$. By Lemma 2.5.6 we have $u_{a+t} u_a^{-1} = u_{e+p}^{-1} m_{o,a+t} m_{t,a}^{-1} u_{e+i+t} = m_{t,a}^{-1} u_{e+i+t}$: also $u_b u_c^{-1} = u_{e+i+t}^{-1} m_{t,b} m_{s,c}^{-1} u_{e+j+s}$: also, $u_d u_{d+s}^{-1} = u_{e+j+s}^{-1} m_{s,d} \times m_{o,d+s}^{-1} u_{e+p} = u_{e+j+s}^{-1} m_{s,d}$. Combining these three results we see that $z = m_{t,a}^{-1} u_{e+i+t} g_i u_{e+i+t}^{-1} m_{t,b} m_{s,c}^{-1} u_{e+j+s} h_j u_{e+j+s}^{-1} m_{s,d}$. Let i+t = rk + p and j+s = wk+p where r,w ∈ N. Suppose first that r ≥ 1, w ≥ 1, then $u_{e+i+t} = (e+p)u^r$ and $u_{e+j+s} = (e+p)u^w$. By Lemma 2.5.5, $u_{e+i+t}\, g_i = (e+p)u^r g_i = (e+p) \times (g_i\alpha_{i,rk-i})u^r$ and $u_{e+j+s}\, h_j = (e+p)u^w h_j = (e+p)(h_j\alpha_{j,wk-j})u^w$. However, by (2.5.3 (iii)), $(e+p)(g_i\alpha_{i,rk-i}) = (g_i\alpha_{i,rk-i})\alpha_{o,p}$ since $g_i\alpha_{i,rk-i} \in G_o$ and $(e+p)(h_j\alpha_{j,wk-j}) = (h_j\alpha_{j,wk-j})\alpha_{o,p}$. By (2.5.3 (i)) we have $(g_i\alpha_{i,rk-i})\alpha_{o,p} = (g_i\alpha_{i,rk-i})\alpha_{rk,p} = (g_i\alpha_{i,rk+p-i})$ and similarly $(h_j\alpha_{j,wk-j})\alpha_{o,p} = (h_j\alpha_{j,wk+p-j})$. Hence $u_{e+i+t}\, g_i = (g_i\alpha_{i,rk+p-i})u^r$, however $g_i\alpha_{i,rk+p-i} \in G_p$ and so $u_{e+i+t}\, g_i = (g_i\alpha_{i,rk+p-i})(e+p)u^r = (g_i\alpha_{i,t})u_{e+rk+p} = (g_i\alpha_{i,t})\, u_{e+i+t}$. Similarly we have $u_{e+j+s} h_j = (h_j\alpha_{j,s})u_{e+j+s}$. With these results, which remain true for r = 0, w = 0, we have that

$$z = m^{-1}_{t,a}\,(g_i\alpha_{i,t})\,u_{e+i+t}\,u^{-1}_{e+i+t}\,m_{t,b}\,m^{-1}_{s,c}\,(h_j\alpha_{j,s})u_{e+j+s}u^{-1}_{e+j+s}\,m_{s,d}$$

$$= m^{-1}_{t,a}\,(g_i\alpha_{i,t})(e+p)\,m_{t,b}\,m^{-1}_{s,c}\,(h_j\alpha_{j,s})(e+p)\,m_{s,d}$$

$$= m^{-1}_{t,a}\,(g_i\alpha_{i,t})\,m_{t,b}\,m^{-1}_{s,c}\,(h_j\alpha_{j,s})\,m_{s,d}\,,\quad \text{since } g_i\alpha_{i,t},\ h_j\alpha_{j,s} \in G_p.$$

This is the same middle term as is obtained when (a,g_i,b) and (c,h_j,d) are multiplied in $S(E,\ T,\ k,\ G_i,\gamma_i,\ e,\ v_f)$ and so we have that $(xy)\psi = (x\psi)(y\psi)$ and ψ is a homomorphism.

2.6 Applications and Special Cases

There are two main lines of approach which make considerable simplifications of the above results and give rise to several already established results. One approach is to simplify E, first by considering a 0-direct union of ω-chains as was done by Lallement in [3], and then by specialising again and examining the case when E is an ω-chain with zero. The other means of refinement is to take $k = 1$ and with this to consider the various cases of E. Before embarking on either of these we investigate the case when, for all $e \in E^*$, v_f is the identity of $G_{\underline{f}+1}$, where multiplication is much simplified.

2.6.1 Definition: A semigroup S, where $\mathcal{H}$ is a congruence on S, is said to 'split over $\mathcal{H}$' if there exists a set of representatives of the $\mathcal{H}$-classes of S which form a subsemigroup of S.

2.6.2 Theorem: (i) The semigroup $S = S(E,\ T,\ k,\ G_i,\ \gamma_i,\ e,\ v_f)$ where, for all $f \in E^*$, v_f is the identity of $G_{\underline{f}+1}$, is a 0-simple inverse semigroup whose semilattice is an ω-tree and which splits over $\mathcal{H}$.

(ii) Conversely, let S be a 0-simple inverse semigroup, whose semilattice is an ω-tree with zero, which splits over $\mathcal{H}$. Then S is of the form $S(E,\ T,\ k,\ G_i,\ \gamma_i,\ e,\ v_f)$ where, for all $f \in E^*$, v_f is the identity of $G_{\underline{f+1}}$.

Proof: (i) From Theorems 2.4.1, 2.4.4 and 2.4.5 we have that S is a 0-simple inverse semigroup whose semilattice is an ω-tree with zero and where $\mathcal{H}$ is a congruence on S. Consider $\mathcal{U} = \{(a,e_i,b) \in S;\ e_i$ is the identity of $G_i\}\cup\{0\}$. Then $\mathcal{U}$ is a subset of S and, from Theorem 2.4.5 (2), is a set of representatives of the $\mathcal{H}$-classes of S. Let $(a,e_i,b),(c,e_j,d) \in \mathcal{U}$. If $bc = 0$, then $(a,e_i,b)(c,e_j,d) = 0 \in \mathcal{U}$. If $bc \neq 0$, then $(a,e_i,b)(c,e_j,d) = (a+t, m_{t,a}^{-1}(e_i\alpha_{i,t})m_{t,b}m_{s,c}^{-1}(e_j\alpha_{j,s}) \times m_{s,d}, d+s)$ where $t = [b,bc]$ and $s = [c,bc]$. However $m_{t,a} = m_{t,b} = m_{s,c} = m_{s,d}$ = the identity element of $G_{\underline{a+t}}$. Also $e_i\alpha_{i,t} = e_{\underline{a+t}} = e_j\alpha_{j,s}$ and so $(a,e_i,b)(c,e_j,d) = (a+t, e_{\underline{a+t}}, d+s) \in \mathcal{U}$. Hence $\mathcal{U}$ is a subsemigroup of S and so S splits over $\mathcal{H}$.

See Appendix (ii) From Theorem 2.5.3, S is of the form $S(E, T, k, G_i, \gamma_i, e, v_f)$. Since S splits over $\mathcal{H}$ there exists a set of representatives A of the $\mathcal{H}$-classes of S which forms a subsemigroup of S. Assume that, in the notation of Theorem 2.5.3, the set of representatives u_f chosen are elements of this subsemigroup of S. Examining, in the light of this, the definition of $m_{t,f}$ we have $m_{t,f}u_{f+t} = u_{e+i+t}u_f$, where $\underline{f} = i$. If $\underline{f+t} = p$ we have $(H_{u_{e+i+t}u_f})\phi = (e+p, e+i+t)(e+i,f) = (e+p, f+t)$ and so, as the set of representatives A forms a subsemigroup of S, we have $u_{e+i+t}u_f = u_{f+t}$. Hence $m_{t,f}u_{f+t} = u_{f+t}$ so that $m_{t,f}$ is the identity of G_p. This means that, for all $f \in E^*$, $v_f = m_{1,f}$ is the identity of $G_{\underline{f+1}}$.

See Appendix 2.6.3 From the above theorem we have a sufficient condition for a semigroup $S(E, T, k, G_i, \gamma_i, e, v_f)$ to split over $\mathcal{H}$: namely that v_f is the identity of $G_{\underline{f+1}}$, for all $f \in E^*$. ~~However a sufficient condition for this to occur is that there exists a set of representatives u_f of the $\mathcal{H}$-classes of S such that for all $f \in E^*$ and all $t \in N$, $u_{e+i+t}u_f = u_{f+t}$ where $\underline{f} = i$.~~

We apply this criterion below in considering the first simplification of E.

2.6.4 A semilattice with zero is said to be a 0-direct union of ω-chains if it is isomorphic to the set $(N \times I) \cup \{0\}$ with the ordering $0 < (n,\alpha)$ for all $n \in N$ and for all $\alpha \in I$, and $(n,\alpha) \leq (m,\beta) \iff \alpha = \beta$ and $n \geq m$, where $n,m \in N$ and $\alpha,\beta \in I$.

2.6.5 <u>Theorem</u>: (i) Let E be a 0-direct union of ω-chains. Then $S = S(E, T, k)$ is a 0-subtransitive inverse subsemigroup of T_E whose semilattice is a 0-direct union of ω-chains.

(ii) Let S be a 0-subtransitive inverse subsemigroup of T_E, where E is an ω-tree with zero, whose semilattice is a 0-direct union of ω-chains. Then E is a 0-direct union of ω-chains and there exist $k \in N$, with $k \geq 1$, and a transversal T of the components of E^* such that $S = S(E, T, k)$.

<u>Proof</u>: (i) That S is a 0-subtransitive inverse subsemigroup of T_E follows immediately from Theorem 2.2.3. From [9, Theorem 3.2 (ii)] we have that S has semilattice isomorphic to E, so that the semilattice of S is a 0-direct union of ω-chains.

(ii) From [9, Theorem 3.2 (ii)] the semilattice S is isomorphic to E. Hence E is a 0-direct union of ω-chains. The remainder of the result holds by Theorem 2.2.5.

In [3] Lallement considers those 0-simple inverse semigroups whose semilattices are 0-direct unions of ω-chains. First he considers the case when $\mathcal{H} = i$. If $\mathcal{H} = i$, then the semigroup is fundamental and by [9, Theorem 3.2 (i)] we have that S is isomorphic to a 0-subtransitive inverse subsemigroup of T_E. Thus the first case considered by Lallement is the same as that in Theorem 2.6.5.

2.6.6 In [3, Proposition 1] Lallement states his result for 0-simple inverse semigroups whose semilattices are 0-direct unions of ω-chains and where $\mathcal{H} = i$.

<u>Proposition 1</u>. Let Λ be a set and $p : \Lambda \times \Lambda \to \mathbf{Z}_d$ be a mapping into the set of integers mod d satisfying $p(\alpha,\beta) + p(\beta,\gamma) = p(\alpha,\gamma)$ and $p(\alpha,\alpha) = 0$, for all $\alpha,\beta,\gamma \in \Lambda$. Let $S(\Lambda, p,d)$ be a set consisting of 0 and the quadruples $(i,j)_{\alpha\beta}$, where $i,j \in N$ and $\alpha,\beta \in \Lambda$, such that $i-j \equiv p(\alpha,\beta) \pmod{d}$. We define a multiplication on $S(\Lambda,p,d)$ such that the only non zero products are $(i,j)_{\alpha\beta}(k,l)_{\beta\gamma} = (i + [k-j], l + [j-k])_{\alpha\gamma}$ where $[n] = n$ if $n \geq 0$ and $[n] = 0$ if $n < 0$. Then $S(\Lambda,p,d)$ is a 0-simple inverse semigroup whose semilattice is a 0-direct union of ω-chains and $\mathcal{H} = i$.

Conversely if S is a 0-simple inverse semigroup whose semilattice is a 0-direct union of ω-chains and $\mathcal{H} = i$ in S then there exist Λ,p,d such that $S \cong S(\Lambda,p,d)$.

2.6.7 We must now reconcile Theorem 2.6.5 with Lallement's proposition quoted in (2.6.6). We note firstly that if E is a 0-direct union of ω-chains then there exists a set Λ such that $E = (N \times \Lambda) \cup \{0\}$ where $(n,\alpha) \leq (m,\beta) \iff \alpha=\beta$ and $n \geq m$ for $n,m \in N$, $\alpha,\beta \in \Lambda$.

Let S be a 0-simple inverse semigroup whose semilattice $E = (N \times \Lambda) \cup \{0\}$ and where $\mathcal{H} = i$. As noted earlier we have by [9, Theorem 3.2 (i)] that S is isomorphic to a 0-subtransitive inverse subsemigroup of T_E. Thus we are in the situation of Theorem 2.6.5 (ii) and also of Theorem 2.2.5. let us identify S with $S(E,T,k)$. Following the notation of Theorem 2.2.5 we make the following definitions:-

Fix $(0,\alpha) \in E^*$. Let $z_{(n,\gamma)} = (0,\gamma)$ for all $n \in N$ and let $e_{(n,\gamma)} = (p_\gamma,\gamma)$ where p_γ is the least element of N such that $((0,\alpha), (p_\gamma,\gamma)) \in S$. Let $T = \{(p_\gamma,\gamma) : \gamma \in \Lambda\}$. We note that $p_\gamma < k$. Suppose that $p_\gamma \geq k$. Then since $((0,\gamma),(k,\gamma)) \in S$ and $((0,\alpha),(p_\gamma,\gamma)) \in S$ we have $((0,\alpha),(p_\gamma,\gamma))((k,\gamma),(o,\gamma)) = ((0,\alpha), (p_\gamma-k,\gamma)) \in S$ which contradicts the definition p_γ.

Using this notation we examine $(\underline{r,\gamma})$. Recalling that $(\underline{r,\gamma})$ is the remainder after division by k of $[(p_\gamma,\gamma),(p_\gamma,\gamma)(r,\gamma)] - [(r,\gamma),(p_\gamma,\gamma)(r,\gamma)]$, there are three cases to consider:- (a) $p_\gamma < r$ (b) $p_\gamma = r$ (c) $p_\gamma > r$. In (a) $(\underline{r,\gamma}) \equiv r-p_\gamma \pmod{k}$, in (b) $(\underline{r,\gamma}) = 0 \equiv r-p_\gamma \pmod{k}$ and in (c) $(\underline{r,\gamma}) \equiv -(p_\gamma-r) \pmod{k} \equiv r-p_\gamma \pmod{k}$. Applying this we have $S = S(E, T, k) =$
$\{((n,\beta),(m,\gamma)) : (n,\beta),(m,\gamma) \in E^* \text{ and } n-p_\beta \equiv m-p_\gamma \pmod{k}\} \cup \{0\}$
$=\{((n,\beta),(m,\gamma)) : n,m \in N, \beta,\gamma \in \Lambda \text{ and } n-m \equiv p_\beta-p_\gamma \pmod{k}\} \cup \{0\}$
This leads us to define $p(\beta,\gamma) = p_\beta-p_\gamma \pmod{k}$ for all $\beta,\gamma \in \Lambda$. It can be quickly checked that $p:\Lambda \times \Lambda \to Z_k$ satisfies the conditions of 2.6.6. Thus we have $S(E, T, k) = \{((n,\beta),(m,\gamma)) : m,n \in N, \beta,\gamma \in \Lambda \text{ and } n-m \equiv p(\beta,\gamma) \pmod{k}\} \cup \{0\}$. Multiplication in $S(E, T, k)$ is now as follows:-

the only non zero products are $((n,\beta),(m,\gamma))((r,\gamma),(q,\delta)) = ((n,\beta) + t, (q,\delta) + s)$, where $t = [(m,\gamma),(m,\gamma)(r,\gamma)]$ and $s = [(r,\gamma),(m,\gamma)(r,\gamma)]$, $= ((n+t,\beta),(q+s,\delta)) = ((n + [r-m],\beta),(q + [m-r],\delta))$. This is exactly the multiplication in $S(\Lambda,p,k)$ and so $S(E, T, k) = S(\Lambda,p,k)$.

We now proceed with the non-fundamental case where E is a 0-direct union of ω-chains,

2.6.8 <u>Theorem</u>: If S is a 0-simple inverse semigroup whose semilattice is a 0-direct union of ω-chains, then S splits over $\mathcal{H}$.

Proof: By Theorem 2.5.1 $\mathcal{H}$ is a congruence on S. Let E = $(N \times \Lambda) \cup \{0\}$ and fix $(0,\alpha) \in E^*$. With the notation of 2.6.7 and Theorem 2.5.3 we select a set of representatives of ~~the~~ certain $\mathcal{H}$-classes of S as follows:- let $u = u_{(k,\alpha)}$ and let $c_\beta = u_{(0,\beta)}$ for all $\beta \in \Lambda$, $\beta \neq \alpha$. We then make the following stipulations:-

(a) $u^n = u_{(nk,\alpha)}$ for all $n \in N$ with $n \geq 1$

(b) $u_{(i,\alpha)}$ is the identity of the group G_i for $i \in N$ with $0 \leq 1 \leq k-1$

(c) $(i,\alpha)u^n = u_{(nk+i,\alpha)}$ for all $i,n \in N$ with $n \geq 1$ and $0 \leq i \leq k-1$

(d) $(i,\alpha)u^{m+1} c_\beta = u_{(mk+p_\beta+i,\beta)}$ for all $m \in N$ and $i \in N$ with $0 \leq i \leq k-1$ and for all $\beta \in \Lambda$ with $\beta \neq \alpha$.

(e) $(k-p_\beta+j,\alpha)c_\beta = u_{(j,\beta)}$ if $0 \leq j < p_\beta$, $\beta \neq \alpha$.

We check briefly that these stipulations are valid. We have $(H_u n)\phi = ((0,\alpha),(k,\alpha))^n = ((0,\alpha),(nk,\alpha))$ also $(H_{(i,\alpha)u^n})\phi = ((i,\alpha),(i,\alpha))((0,\alpha),(nk,\alpha)) = ((i,\alpha),(nk+i,\alpha))$. We note that $(H_{c_\beta})\phi = ((k-p_\beta,\alpha),(0,\beta))$ and so $(H_{(i,\alpha)u^{m+1}c_\beta})\phi = ((i,\alpha),(i,\alpha))((0,\alpha),((m+1)k,\alpha))((k-p_\beta,\alpha),(0,\beta))$
$= ((i,\alpha),(i,\alpha))((0,\alpha),((mk+p_\beta),\beta)) = ((i,\alpha),((mk+i+p_\beta),\beta))$.

We now check that this set of representatives satisfies the condition stated in 2.6.3. First we examine $u_{(i+t,\alpha)}\, u_{(nk+i+p_\beta,\beta)}$ where $\beta \neq \alpha$. Let $i+t = sk+p$ where $0 \leq p \leq k-1$ and we have $u_{(i+t,\alpha)}\, u_{(nk+i+p_\beta,\beta)} = (p,\alpha)\, u^s\, (i,\alpha)\, u^{n+1}\, c_\beta$. There are two cases to consider:- the first that $s = 0$ and the second that $s \geq 1$.

(i) If $s = 0$ then $i+t = p$ and we have $u_{(i+t,\alpha)}\, u_{(nk+i+p_\beta,\beta)} = (p,\alpha)(i,\alpha)\, u^{n+1}\, c_\beta$. However $p \geq i$ and so $u_{(i+t,\alpha)}\, u_{(nk+i+p_\beta,\beta)} = (p,\alpha)\, u^{n+1}\, c_\beta = u_{(nk+p_\beta+p,\beta)} = u_{(nk+p_\beta+i+t,\beta)}$.

(ii) If $s \geq 1$ then $u_{(i+t,\alpha)}\, u_{(nk+i+p_\beta,\beta)} = (p,\alpha)\, u^s\, u^{n+1}\, c_\beta$ as $u(i,\alpha) = u$ and so we have $u_{(i+t,\alpha)}\, u_{(nk+i+p_\beta,\beta)} = (p,\alpha)\, u^{s+n+1}\, c_\beta$
$= u_{((s+n)k+p_\beta+p,\beta)} = u_{(nk+i+t+p_\beta,\beta)}$.

In both cases we have the required result. A similar argument shows that if $0 \leq j < p_\beta$, $\beta \neq \alpha$, then $u_{(i+t,\alpha)} u_{(j,\beta)} = u_{(j+t,\beta)}$ where $i = k - p_\beta + j$. If however we consider $u_{(i+t,\alpha)} u_{(nk+i,\alpha)}$ where $i+t = sk+p$ we have $u_{(i+t,\alpha)} u_{(nk+i,\alpha)} = (p,\alpha) u^s (i,\alpha) u^n$ and considering in turn the cases when $s = 0$ and $n = 0$ we have $u_{(i+t,\alpha)} u_{(nk+i,\alpha)} = u_{(nk+i+t,\alpha)}$ which is again the required result.

2.6.9 Theorem: (i) Let E be a 0-direct union of ω-chains. Then $S = S(E, T, k, G_i, \gamma_i, e, v_f)$ where, for all $f \in E^*$, v_f is the identity of $G_{\underline{f}+1}$, is a 0-simple inverse semigroup whose semilattice is a 0-direct union of ω-chains.

(ii) Let S be a 0-simple inverse semigroup whose semilattice is a 0-direct union of ω-chains. Then S is of the form $S(E, T, k, G_i, \gamma_i, e, v_f)$ where v_f is the identity of $G_{\underline{f}+1}$ for all $f \in E^*$.

These results are immediate from Theorem 2.6.8 and Theorem 2.6.2.

2.6.10 In [3,Theorem 2] Lallement states his result for this case. It is as follows:-

Theorem 2: Let $G_0 \xrightarrow{\gamma_0} G_1 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_{d-2}} G_{d-1} \xrightarrow{\gamma_{d-1}} G_0$ be a cycle of group homomorphisms. Let $\alpha_{m,n} = \gamma_m \gamma_{m+1} \cdots \gamma_{n-1}$ where $m,n \in N$ with $m \leq n$. Let $\Sigma = S(\Lambda, p, d, G_i, \gamma_i)$ be the set consisting of 0 and the elements of the form $(i, g_r, j)_{\beta\gamma}$ where $(i,j)_{\beta\gamma} \in S(\Lambda, p, d)$ and $g_r \in G_r$ with $r \equiv (i-p(\beta,\alpha)) (\mathrm{mod}\ d)$ (where α is a fixed element of Λ). On Σ we define a multiplication where the only non zero products are given by $(i, g_r, j)_{\beta\gamma} (k, g_s, l)_{\gamma\delta} = (i+[k-j], g_r \alpha_{u,w} \times g_s \alpha_{v,w}, l + [j-k])_{\beta\delta}$ where $u = j-p(\gamma,\alpha)$, $v = k-p(\gamma,\alpha)$ and $w = \max\{u,v\}$. With this multiplication Σ is a 0-simple inverse semigroup whose semilattice is a 0-direct union of ω-chains.

Conversely every semigroup S which is a 0-simple inverse semigroup whose semilattice is a 0-direct union of ω-chains is of this form.

Clearly we must now reconcile Lallement's theorem quoted in 2.6.10 and Theorem 2.6.9.

2.6.11 With the notation as developed in 2.6.7 we have that if $S = S(E, T, k, G_i, \gamma_i, e, v_f)$ is as defined in Theorem 2.6.9 then

$$S = \{((r,\beta), g_i, (s,\gamma)) : r,s \in N, \beta,\gamma \in \Lambda, r-p_\beta \equiv s-p_\gamma \equiv i \pmod{k} \text{ and } g_i \in G_i\} \cup \{0\}$$

$$= \{((r,\beta), g_i, (s,\gamma)) : r,s \in N, \beta,\gamma \in \Lambda, r-p(\beta,\alpha) \equiv s-p(\gamma,\alpha) \equiv i \pmod{k} \text{ and } g_i \in G_i\} \cup \{0\}, \text{ since } p_\alpha = 0.$$

The set S is thus the same as the set Σ with k = d and we need only now check that the multiplications are the same. We note firstly that $\alpha_{m,n}$ as defined in Σ is the same as $\alpha_{m,n-m}$ in S. Using the result that $S(E, T, k) = S(\Lambda, p, k)$ we have the only non zero products in S are

$((r,\beta), g_i\ (s,\gamma))((n,\gamma), h_j, (q,\delta)) = ((r + [n-s],\beta), g_i\alpha_{i,[n-s]}h_j\alpha_{j,[s-n]}, (q + [s-n],\delta))$.

Multiplication in Σ states that $(r, g_i, s)_{\beta\gamma}\ (n, h_j, q)_{\gamma\delta}$ is a non zero product equal to $(r + [n-s], g_i\alpha_{u,w}\ h_j\alpha_{v,w}, q+[s-n])_{\beta\delta}$ where $u = s-p(\gamma,\alpha)$, $v = n-p(\gamma,\alpha)$, $w = \max\{u,v\}$. Using the information that $\alpha_{u,w}\ (\Sigma) = \alpha_{u,w-u}\ (S)$ we have this product equal to $(r + [n-s], g_i\alpha_{u,w-u}\ h_j\ \alpha_{v,w-v}, q + [s-n])_{\beta\delta}$, where $u = s-p_\gamma = s-p(\gamma,\alpha)$, $v = n-p_\gamma = n-p(\gamma,\alpha)$ and $w = \max\{u,v\}$. Since $u \equiv i \pmod{k}$ and $v \equiv j \pmod{k}$ we have $g_i\ \alpha_{u,w-u} = g_i\ \alpha_{i,w-u}$ and $h_j\ \alpha_{v,w-v} = h_j\ \alpha_{j,w-v}$. However $w-u = [n-s]$ and $w-v = [s-n]$ so that we have the same form for the products in Σ and S.

The next case to consider is the one where E is an ω-chain with zero. This isin fact a special case of the above piece of work, where E is a 0-direct union of ω-chains, this being the case where $|\Lambda| = 1$ and so $E \cong \{e_i : i \in N \text{ and } e_i > e_j \Leftrightarrow i < j\} \cup \{0\}$.

2.6.12 Theorem (i) Let E be an ω-chain with zero. Then $S(E, T, k, G_i, \gamma_i, e, v_f)$ where, for all $f \in E^*$, v_f is the identity of $G_{\underline{f}+1}$, is a 0-simple inverse semigroup whose semilattice is an ω-chain with zero.

(ii) Let S be a 0-simple inverse semigroup whose semilattice E is an ω-chain with zero. Then S is of the form $S(E, T, k, G_i, \gamma_i, e, v_f)$ where, for all $f \in E^*$, v_f is the identity of $G_{\underline{f}+1}$.

Proof: (i) This follows immediately from Theorems 2.4.1, 2.4.4 and 2.4.5.

(ii) Since an ω-chain is, trivially, a 0-direct union of ω-chains we apply Theorem 2.6.9 (ii) and immediately have the result.

In [7] Munn obtained an apparently different structure theorem for the same type of semigroup as described in (2.6.12). His results are stated in [7, Theorem 3.3 and Theorem 4.11]. We now show the results to be equivalent.

2.6.13 Let $\tau: S(E, T, k, G_i, \gamma_i, e, v_f) \rightarrow S(k, G_i, \gamma_i) \cup \{0\}$, where $S(E, T, k, G_i, \gamma_i, e, v_f)$ is as described in 2.6.12 with T consisting of the greatest element of E, and $S(k, G_i, \gamma_i)$ is as in [7, Theorem 3.3], be defined as follows:-

$$(e_{rk+i}, g_i, e_{sk+i})\tau = (r, g_i, s)$$
$$0\tau = 0$$

It can be easily checked that τ is an isomorphism and so the two structure theorems are equivalent.

2.6.14. When E is an ω-chain with zero and $S = S(E, T, k, G_i, \gamma_i, e, v_f)$ is as in Theorem 2.6.12 then we note that there are no zero products in $S \setminus \{0\}$. Hence $S \setminus \{0\}$ is a simple regular ω-semigroup in the terminology of [7]. Thus $S \setminus \{0\} \cong S(k, G_i, \gamma_i)$ as described in [7].

If we now return to the original situation, where E is an ω-tree with zero, we can begin a different set of specialisations by taking $k = 1$, i.e. by having a semigroup with one non zero $\mathcal{D}$-class, in other words a 0-bisimple inverse semigroup with semilattice an ω-tree with zero.

2.6.15 Theorem: (i) Let E be an ω-tree with zero and $k = 1$. Then $S(E, T, k, G, \gamma, e, v_f)$ is a 0-bisimple inverse semigroup whose semilattice is an ω-tree with zero.

(ii) Let S be a 0-bisimple inverse semigroup whose semilattice E is an ω-tree with zero. Then S has the form $S(E, T, k, G_i, \gamma_i, e, v_f)$ where $k = 1$.

Proof: In the case $k = 1$, there is one group G and one homomorphism $\gamma: G \to G$. With the original notation of (2.3.3), $\alpha_{i,t} = \gamma^t$, for $t \in N$, $t \geq 1$.

(i) This is immediate from Theorems 2.4.1, 2.4.4 and 2.4.5.

(ii) This is immediate from Theorems 2.5.3 and 2.4.5.

Notice that in this case $S(E, T, k, G, \gamma, e, v_f) = \{(a,g,b): a,b \in E^*, g \in G\} \cup \{0\}$ where multiplication is as follows:-
the only non zero products are $(a,g,b)(c,h,d)$ where $bc \neq 0$ and $(a,g,b)(c,h,d) = (a+t, m_{t,a}^{-1}(g\gamma^t)m_{t,b}\, m_{s,c}^{-1}(h\gamma^s)m_{s,d}, d+s)$ where $t = [b,bc]$ and $s = [c,bc]$. This is the same result as is stated in [5, Theorem 6.1].

2.6.16 Theorem (i) Let E be a 0-direct union of ω-chains and let $k = 1$. Then $S(E, T, k, G, \gamma, e, v_f)$ where, for all $f \in E^*$, v_f is the identity of G, is a 0-bisimple inverse semigroup whose semilattice is a 0-direct union of ω-chains.

(ii) Let S be a 0-bisimple inverse semigroup whose semilattice E is a 0-direct union of ω-chains. Then S is of the form $S(E, T, k, G_i, \gamma_i, e, v_f)$ where $k = 1$ and, for all $f \in E^*$, v_f is the identity of $G_{\underline{f}+1}$.

Proof: (i) This follows from Theorems 2.6.15 (i) and 2.4.5.

(ii) From Theorem 2.6.8 we have that S splits over $\mathcal{H}$ and so by Theorem 2.6.2 (ii) we have that S is of the form $S(E, T, k, G_i, \gamma_i, e, v_f)$ where, for all $f \in E^*$, v_f is the identity of $G_{\underline{f+1}}$. If we now apply Theorem 2.6.15 (ii) we have the result.

2.6.17. Note that when $k = 1$ and, for all $f \in E^*$, v_f is the identity of G, if we take $E = (N \times \Lambda) \cup \{0\}$ then $S = S(E, T, k, G, \gamma, e, v_f) = \{((m,\alpha), g, (n,\beta)) : m,n \in N, \alpha,\beta \in \Lambda, g \in G\} \cup \{0\}$.
The multiplication on S is such that the only non zero products are $((m,\alpha),g,(n,\beta))((p,\beta), h,(q,\delta)) = ((m+t,\alpha), (g\gamma^t)(h\gamma^s), (q+s,\delta))$ where $t = [(n,\beta),(n,\beta)(p,\beta)]$ and $s = [(p,\beta),(n,\beta)(p,\beta)]$. Let $v = \max(n,p)$ then $t = v-n$ and $s = v-p$. Hence $((m,\alpha),g,(n,\beta))((p,\beta) h,(q,\delta)) = ((m-n+v,\alpha),(g\gamma^{v-n})(h\gamma^{v-p}), (q-p+v,\delta))$.

Thus we have that the result stated in Theorem 2.6.16 is exactly that of [10, Theorem 4.2].

2.6.18 Theorem: (i) Let E be an ω-chain with zero and $k = 1$. Then $S(E, T, k, G, \gamma, e, v_f)$, where for all $f \in E^*$, v_f is the identity of G, is a 0-bisimple inverse semigroup whose semilattice is an ω-chain with zero.

(ii) Let S be a 0-bisimple inverse semigroup whose semilattice E is an ω-chain with zero. Then S is of the form $S(E, T, k, G_i, \gamma_i, e, v_f)$ where $k = 1$ and for all $f \in E^*$, v_f is the identity element of $G_{\underline{f+1}}$.

Proof: This result follows immediately from Theorem 2.6.16 as an ω-chain with zero is, trivially, a 0-direct union of ω-chains.

2.6.19 Applying (2.6.17) to the case when E is an ω-chain with zero we have $S = S(E, T, k, G, \gamma, e, v_f) = \{(m, g, n) : m,n \in N, g \in G\} \cup \{0\}$ with multiplication as follows:- the only non zero products in S are $(m,g,n)(p,h,q) = (m-n+t,(g\gamma^{t-n})(h\gamma^{t-p}),q-p+t)$ where $t = \max(n,p)$.

Hence we have that $S = \{(m,g,n): m,n \in N,\ g \in G\} \cup \{0\}$,

where G is a group, with multiplication defined as follows:-

$(m,g,n)(p,h,q) = (m-n+t,\ (g\gamma^{t-n})(h\gamma^{t-p}),\ q-p+t)$, where $t = \max(n,p)$

and $\gamma: G \to G$ is an endomorphism, and all other products are zero,

is a 0-bisimple inverse semigroup whose semilattice is an ω-chain with zero

and that, conversely, every 0-bisimple inverse semigroup whose semilattice

is an ω-chain with zero is of this form.

From this we can readily deduce that if G is a group and

$\gamma: G \to G$ is an endomorphism then $B = \{(m,g,n): m,n \in N,\ g \in G\}$ with

multiplication as follows:-

$(m,g,n)(p,h,q) = (m-n+t,\ (g\gamma^{t-n})(h\gamma^{t-p}),\ q-p+t)$ where $t = \max(n,p)$,

is a bisimple inverse semigroup whose semilattice is an ω-chain.

Conversely, every bisimple inverse semigroup whose semilattice is

an ω-chain is of this form. This is exactly the result obtained

by Reilly in [12].

It should be noted that in the above situation, where E is

an ω-chain with zero, the parameter T is is redundant, there being

only one component of E*.

Apart from these two paths of specialisation through first

the semilattice and then the assumption that k = 1 we can consider

an independent specialisation of E and the ensuing case with k = 1.

For this example we let $E = \{e_n : n \in I$ and $e_n \leqq e_m \iff n \geq m\}$ where

I denotes the set of integers.

2.6.20 <u>Theorem</u>: (i) The semigroup $S = S(E, T, k, G_i, \gamma_i, e_o, v_{e_n}) \setminus \{0\}$,

where v_{e_n} is the identity of $G_{\underline{e+1}}$ for all $n \geq 0$, is a simple inverse

semigroup whose semilattice is isomorphic to the integers under the

reverse of the normal ordering.

(ii) If S is a simple inverse semigroup whose

semilattice E is isomorphic to the set of integers under the reverse of the normal ordering then S is of the form $S(E, T, k, G_i, \gamma_i, e_o, v_{e_n}) \setminus \{0\}$ where v_{e_n} is the identity element of $G_{\underline{e}_n+1}$ for all $n \geq 0$.

Proof: (i) This follows from Theorems 2.4.1, 2.4.4 and 2.4.5 since no products of the form $(e_n, g_i, e_n)(e_p, h_j, e_q)$ are zero.

(ii) From Theorem 2.5.3, S with a 0 adjoined is of the form $S(E, T, k, G_i, \gamma_i, e_n, v_{e_m})$. The set T is here a singleton since there is one component only of E*. We therefore take $T = \{e_o\}$ and e_o to be the fixed element used as a parameter.

With the notation of Theorem 2.5.3 we select a set of representatives of the non zero $\mathcal{H}$-classes of S with the following stipulations:-

let $u = u_{e_k}$ and $u^n = u_{e_{nk}}$ for all $n \in N$, $n \geq 1$.

let $e_p u^n = u_{e_{p+nk}}$ where $n,p \in N$ and $0 \leq p \leq k-1$ and $n \geq 1$.

let $e_i = u_{e_i}$ where $i \in N$ and $0 \leq i \leq k-1$.

We now examine, in the light of the above specifications for the set of representatives, the elements v_{e_n} where $n \geq 0$. By definition v_{e_n} is such that $v_{e_n} u_{\underline{e}_n+1} = u_{e_o+i+1} u_{e_n}$ where $n \equiv i \pmod k$ and $0 \leq i \leq k-1$. By the specifications above $u_{e_o+i+1}\, u_{e_n} = u_{e_o+i+1}\, u_{e_{i+sk}}$, where $n = i+sk$, $i,s \in N$, and $0 \leq i \leq k-1$, so that $u_{e_o+i+1}\, u_{e_{i+sk}}$ $= u_{e_{i+1}}\, e_i u^s$. If $i \leq k-2$ then $u_{e_{i+1}}\, e_i u^s = e_{i+1}\, e_i u^s = e_{i+1}\, u^s =$ $u_{e_{i+1+sk}}$ and if $i = k-1$ $u_{e_{i+1}}\, e_i u^s = u e_i u^s = u u^s = u^{s+1} = u_{e_{(s+1)k}}$.
In both cases $u_{e_o+i+1}\, u_{e_n} = u_{e_n+1}$ and so v_{e_n} is the identity of the group $G_{\underline{e}_n+1}$.

If we recall from the proof of (i) that no products of elements of $S(E, T, k, G_i, \gamma_i, e_o, v_n) \setminus \{0\}$ are zero then we have the result.

2.6.21. We note that in the case of Theorem 2.6.20 (ii) S is of the form $\{(e_n,g_i,e_m): m,n,i \in I$ with $0 \le i \le k-1,\ n \equiv m \equiv i \pmod{k},\ g_i \in G_i\}$ with multiplication as follows:-

$$(e_n,g_i,e_m)(e_p,h_j,e_q) = (e_n,g_i\, m_{s,e_p}^{-1}(h_j\alpha_{j,s})\, m_{s,e_q},\ e_{q+s}) \text{ if } m \ge p \text{ and } s = m-p$$

$$= (e_{n+t},\ m_{t,e_n}^{-1}(g_i\alpha_{i,t})\, m_{t,e_m} h_j, e_q) \text{ if } m \le p \text{ and } t = p-m$$

With this representation for S put k = 1 and we have the following:-

2.6.22 <u>Theorem</u>: (i) Let G be a group and α an endomorphism of G with α^0 the identity automorphism on G. For each $n \in I$ chose $u_n \in G$ such that u_n is the identity of G if $n \geq 0$. Define $v_n = u_{n+1}$ for all $n \in I$. For $t \in N$ and $n \in I$ let $m_{t,n} = (v_n\, \alpha^{t-1})(v_{n+1}\, \alpha^{t-2})\ldots \times v_{n+t-1}$ where $t \ge 1$ and $m_{o,n}$ be the identity of G. Then, if $S = \{(e_p,g,e_q): p,q \in I,\ g \in G\}$ with multiplication defined as in (2.6.21), S is a bisimple inverse semigroup whose semilattice is isomorphic to I under the reverse of the normal ordering.

(ii) Conversely, if S is a bisimple inverse semigroup whose semilattice is isomorphic to I with the reverse of the normal ordering then S has the form described in (i).

<u>Proof</u>: (i) This follows from Theorem 2.6.20 (i) noting that, since k = 1, S has only one $\mathcal{D}$-class and so is bisimple.

(ii) This follows from Theorem 2.6.20 (ii) and (2.6.21).

This is exactly the result obtained by Warne in [14, Theorem 1.3].

3. A 0-simple inverse semigroup whose semilattice admits a factorisation compatible with its $\mathcal{D}$-structure

In [5] McAlister gives a structure theorem for 0-bisimple inverse semigroups in terms of groups and 0-uniform semilattices. In this chapter we extend this to a structure theorem for a particular type of 0-simple inverse semigroup. Firstly, however, we require a summary of some points in McAlister's paper.

3.1 Introduction

3.1.1 Let E be a 0-uniform semilattice. Then, following the pattern of [5, Section 2], we define an addition on E. Fix an element $k \in E^*$ and let $E^+ = \{x \in E^*: x \leq k\}$. For each $e \in E^*$ let I_e denote the set of isomorphisms from E^+ onto $\{x \in E^*: x \leq e\}$: $I_e \neq \phi$ since E is a 0-uniform semilattice. By an addition on E with identity k we mean a choice function Φ on $\{I_e: e \in E^*\}$ such that $\Phi(k)$ is the identity on E^+. If Φ is an addition on E and $e \in E^+$ and $f \in E^*$ we write $e+f$ for $e\Phi(f)$; if $g,h \in E^*$ with $g \leq h$ we write $g-h$ for $g(\Phi(h))^{-1}$. The addition Φ is associative if $(e+f)+g = e+(f+g)$ where $e,f \in E^+$, $g \in E^*$.

3.1.2 For completeness we include the statement of McAlister's structure theorem, [5, Theorem 3.2].

<u>Theorem</u>: Let E be a 0-uniform semilattice and let Θ be a fixed non zero element of E; let $E^* = E \setminus \{0\}$ and $E^+ = \{x \in E^*: x \leq \Theta\}$. Let Φ be an addition on E with identity Θ and let G be a group, with identity element 1, which acts on E^+ by (order) automorphisms. Suppose that functions $f: E^+ \times E^* \to G$ and $[\ ,\ ]: E^+ \times G \to G$ are given which satisfy:-

1. $f(\theta,b) = 1 = f(a,\theta)$ for each $a \in E^+$, $b \in E^*$.

2. $[\theta,g] = g$ for each $g \in G$.

3. $f(a,b)f(a+b,c) = [a,f(b,c)]f(af(b,c),b+c)$

   $f(a,b)[a+b,k] = [a,[b,k]]f(a[b,k],bk)$

   for all $a,b \in E^+$, $c \in E^*$, $k \in G$.

4. $[a,g][ag,h] = [a,gh]$ for each $a \in E^+$, $g,h \in G$.

5. $(a+b)k = a[b,k] + bk$ for all $a,b \in E^+$, $k \in G$.

6. $(a+b)+c = af(b,c) + (b+c)$ for all $a,b \in E^+$, $c \in E^*$.

where the group action is denoted by juxtaposition.

Then the set $E^* \times G \times E^*$, together with zero, forms a 0-bisimple inverse semigroup $B^o(E,\theta, \Phi; G, o; f, [\ ,\ ])$ under the multiplication

$$(a,g,b)(c,h,d) = \begin{cases} ((bc-b)g^{-1} + a,\ uv,\ (bc-c)h+d) & \text{if } bc \neq 0 \\ 0 & \text{, otherwise} \end{cases}$$

$$(a,g,b)0 = 0 = 0(a,g,b) = 0^2,$$

where $u = (f((bc-b)g^{-1},a))^{-1}\ [(bc-b)g^{-1},g]\ f(bc-b,b)$

and $v = (f(bc-c,c))^{-1}\ [bc-c,h]\ f((bc-c)h,d)$.

The group of units is isomorphic to G and the semilattice of idempotents is isomorphic to E.

Conversely, if S is a 0-bisimple inverse semigroup with semilattice of idempotents isomorphic to E and group of units isomorphic to G, then $S \cong B^o(E,\theta, \Phi; G, o; f, [\ ,\ ])$ for some addition Φ on E with identity θ, action o of G on E^+ and functions f, $[\ ,\ ]$ for which 1-6 (above) hold.

3.1.3 <u>Corollary</u>: Let E be a 0-uniform semilattice and let θ be a fixed non zero element of E; let $E^* = E\setminus\{0\}$ and $E^+ = \{x \in E^*: x \leq \theta\}$. Let Φ be an addition on E, with identity θ, which is associative, and let G be a group with identity 1, acting trivially on E^+. Suppose that functions $f:E^+ \times E^* \to G$ and $[\ ,\ ]:E^+ \times G \to G$ are given which satisfy:-

1. $f(\Theta,b) = 1 = f(a,\Theta)$ for each $a \in E^+$, $b \in E^*$

2. $[\Theta,g] = g$ for each $g \in G$.

3. $f(a,b)f(a+b,c) = [a,f(b,c)]f(a,b+c)$

   $f(a,b)[a+b,k] = [a,[b,k]]f(a,b)$

   for all $a,b \in E^+$, $c \in E^*$, $k \in G$.

4. $[a,g][a,h] = [a,gh]$ for each $a \in E^+$, $g,h \in G$.

Then the set $E^* \times G \times E^*$, together with zero, forms a 0-bisimple inverse semigroup $B^0(E, \Theta, \Phi; G; f,[\ ,\ ])$ under the multiplication

$$(a,g,b)(c,h,d) = \begin{cases} ((bc-b)+a,uv,(bc-c)+d) & \text{if } bc \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$(a,g,b)0 = 0 = 0(a,g,b) = 0^2$

where $u = (f(bc-b,a))^{-1}[bc-b,g]f(bc-b,b)$

and $v = (f(bc-c,c))^{-1}[bc-c,h]f(bc-c,d)$.

The group of units is isomorphic to G and the semilattice of idempotents is isomorphic to E.

3.2 The construction of the groupoid $S(E, k, \Phi; A, \tau,\Theta)$

In this section we describe a process for constructing a 0-simple inverse semigroup from a certain type of 0-uniform semilattice and a monoid by a method based on that of McAlister described in (3.1).

3.2.1 Let E be a 0-uniform semilattice and let k be a fixed element of E^*; let $E^+ = \{x \in E^* : x \leq k\}$. Let Φ be an associative addition on E with identity k.

Let A be a monoid with identity element 1.

Suppose that a function $\tau : E^+ \times E^* \to H_1$ (of A) is given and, for all $e \in E^+$, an endomorphism of A, Θ_e, is defined such that Θ_k is the identity on A and $(A)\Theta_e \subsetneq H_1$ if $e \neq k$ and also the following conditions are satisfied:-

3.2.1 (1) $\tau(k,f) = 1 = \tau(e,k)$ for each $e \in E^+$, $f \in E^*$.

3.2.1 (2) $\tau(e,f)\ \tau(e+f,g) = (\tau(f,g))\theta_e\ \tau(e,f+g)$ for each $e,f \in E^+$, $g \in E^*$

3.2.1 (3) $\tau(e,f)a\theta_{e+f} = (a\theta_f)\theta_e\tau(e,f)$ for each $e,f \in E^+$, $a \in A$.

Define a multiplication on $E^* \times A \times E^* \cup \{0\}$ as follows:-

$(e,a,f)(g,b,h) = ((fg-f)+e,\ uv,\ (fg-g)+h)$, if $fg \neq 0$, where

$u = (\tau(fg-f,e))^{-1}\ a\theta_{fg-f}\tau(fg-f,f)$ and $v = (\tau(fg-g,g))^{-1}\ b\theta_{fg-g}\ \tau(fg-g,h)$

All other products are zero.

3.2.2 It is readily seen that this multip lication is closed. Clearly if $(e,a,f),(g,b,h) \in E^* \times A \times E^*$ with $fg \neq 0$ then $(fg-f)+e$, $(fg-g)+h$ $\in E^*$ and $u,v \in A$ so that $uv \in A$.

3.2.3 We denote the groupoid formed in (3.2.1) by $S(E,\ k,\ \Phi;\ A,\tau\ ,\theta)$, where θ is the mapping $e \mapsto \theta_e$ $(e \in E^+)$

3.3 $S(E,\ k,\ \Phi;\ A,\ \tau,\theta)$

In this section we establish that the groupoid $S(E,\ k,\ \Phi;\ A,\tau\ ,\theta)$ is a semigroup and state necessary and sufficient conditions for it to be 0-simple. We then examine in detail the structure of $S(E,\ k,\ \Phi;\ A,\ \tau,\theta)$.

3.3.1 <u>Theorem:</u> $S = S(E,\ k,\ \Phi;\ A,\ \tau,\theta)$ is a semigroup with zero.

<u>Proof:</u> Let (e,a,f), (g,b,h), $(l,c,m) \in S\setminus\{0\}$

(a) If $fg = 0 = hl$ then $[(e,a,f)(g,b,h)](l,c,m) = 0(l,c,m) = 0$ and $(e,a,f)[(g,b,h)(l,c,m)] = (e,a,f)0 = 0$.

(b) If $fg = 0$ and $hl \neq 0$ then $[(e,a,f)(g,b,h)](l,c,m) = 0(l,c,m) = 0$. On the other hand, $(e,a,f)[(g,b,h)(l,c,m)] = (e,a,f)((hl-h)+g,x,(hl-l) + m)$ where x is the appropriate middle term. However $(hl-h) + g \leq g$ and so $f((hl-h) +g) = fg((hl-h)+g) = 0$ and $(e,a,f)[(g,b,h)(l,c,m)] = 0$.

(c) If $fg \neq 0$ and $hl = 0$ we can show in a similar manner to (b) that $[(e,a,f)(g,b,h)](l,c,m) = 0 = (e,a,f)[(g,b,h)(l,c,m)]$.

(d) We are now left to consider the case when $fg \neq 0$ and $hl \neq 0$. Here $[(e,a,f)(g,b,h)](l,c,m) = ((fg-f)+e, uv, (fg-g)+h)(l,c,m)$ where $u = (\tau(fg-f,e))^{-1} a\theta_{fg-f}\tau(fg-f,f)$ and $v = (\tau(fg-g,g))^{-1} b\theta_{fg-g}\tau(fg-g,h)$. Let $n = (fg-f)+e$ and $p = (fg-g)+h$, Then $[(e,a,f)(g,b,h)](l,c,m) = (n,uv,p)(l,c,m) = ((pl-p)+n, wx, (pl-l)+m)$ where

$w = (\tau(pl-p,n))^{-1}(uv)\theta_{pl-p}\tau(pl-p,p)$ and $x = (\tau(pl-l,l))^{-1} c\theta_{pl-l}\tau(pl-l,m)$

On the other hand, $(e,a,f)[(g,b,h)(l,c,m)] = (e,a,f)((hl-h)+g, st, (hl-l)+m)$ where $s = (\tau(hl-h,g))^{-1} b\theta_{hl-h}\tau(hl-h,h)$ and $t = (\tau(hl-l,l))^{-1} c\theta_{hl-l}\tau(hl-l,m)$ Let $q = (hl-h)+g$ and $r = (hl-l)+m$, Then $(e,a,f)[(g,b,h)(l,c,m)] = (e,a,f)(q,st,r) = ((fq-f)+e, yz, (fq-q)+r)$ where

$y = (\tau(fq-f,e))^{-1} a\theta_{fq-f}\tau(fq-f,f)$ and $z = (\tau(fq-q,q))^{-1}(st)\theta_{fq-q}\tau(fq-q,r)$.

The outer terms in each of these products are exactly those obtained as outer terms in the semigroup of Corollary 3.1.3. Since associativity has been proved in this case, we can say here that the outer terms in the products $[(e,a,f)(g,b,h)](l,c,m)$ and $(e,a,f)[(g,b,h)\times(l,c,m)]$ are equal. We must now prove that the middle term in each of these products is the same.

From the equality of the outer terms we have the following:-

$(pl-p)+n = (fq-f)+e$ and $(pl-l)+m = (fq-q)+r$.

Hence we have $(pl-p) + ((fg-f)+e) = (fq-f)+e$ and so, as the addition is associative, $((pl-p)+(fg-f))+e = (fq-f)+e$ so that

$$(pl-p)+(fg-f) = fq-f \quad \ldots\ldots \quad 3.3.1\ (i)$$

Operating on both sides of 3.3.1 (i) by f and using again that the addition is associative, we have

$$(pl-p)+fg = fq \quad \ldots\ldots \quad 3.3.1\ (ii)$$

By similar consideration of $(pl-l)+m$ and $(fq-q)+r$ we have

$$pl-l = (fq-q)+(hl-l) \quad \ldots\ldots \quad 3.3.1(iii)$$

and

$$pl = (fq-q)+hl \quad \ldots\ldots \quad 3.3.1(iv)$$

First we examine the middle term wx. We have

$$wx = (\tau(pl-p,n))^{-1}(uv)\theta_{pl-p}\tau(pl-p,p)(\tau(pl-1,1))^{-1}\, c\theta_{pl-1}\tau(pl-1,m).$$

However, θ_{pl-p} is an endomorphism of A and so $(uv)\theta_{pl-p} = (u\theta_{pl-p}) \times (v\theta_{pl-p})$. Now,

$$u\theta_{pl-p} = [(\tau(fg-f,e))^{-1}\, a\theta_{fg-f}\tau(fg-f,f)]\theta_{pl-p}$$
$$= ((\tau(fg-f,e))\theta_{pl-p})^{-1}\,(a\theta_{fg-f})\theta_{pl-p}\,(\tau(fg-f,f))\theta_{pl-p}$$ as θ_{pl-p}

is an endomorphism. However, by 3.2.1(2),

$$(\tau(fg-f,e))\theta_{pl-p} = \tau(pl-p,fg-f)\tau((pl-p)+(fg-f),e)(\tau(pl-p,(fg-f)+e))^{-1}$$
$$= \tau(pl-p,fg-f)\tau(fq-f,e)(\tau(pl-p,n))^{-1}$$, by 3.3.1(i), and

$$(\tau(fg-f,f))\theta_{pl-p} = \tau(pl-p,fg-f)\tau(fq-f,f)(\tau(pl-p,fg))^{-1}$$, by 3.3.1(i).

Also, by 3.2.1(3), $(a\theta_{fg-f})\theta_{pl-p} = \tau(pl-p,fg-f)\; a\theta_{(pl-p)+(fg-f)}(\tau(pl-p,fg-f))^{-1}$

$= \tau(pl-p,fg-f)\; a\theta_{fq-f}(\tau(pl-p,fg-f))^{-1}$ by 3.3.1(ii).

Thus $u\theta_{pl-p} = \tau(pl-p,n)(\tau(fq-f,e))^{-1}\, a\theta_{fq-f}\tau(fq-f,f)(\tau(pl-p,fg))^{-1}$.

However, we also have $v\theta_{pl-p} = (\tau(fg-g,g)\theta_{pl-p})^{-1}(b\theta_{fg-g})\theta_{pl-p}\tau(fg-g,h)\theta_{pl-p}$ since θ_{pl-p} is an endomorphism of A. By 3.2.1 (2) we have

$$\tau(fg-g,g)\theta_{pl-p} = \tau(pl-p,fg-g)\tau((pl-p)+(fg-g),g)(\tau(pl-p,(fg-g)+g))^{-1}$$
$$= \tau(pl-p,fg-g)\tau(fq-g,g)(\tau(pl-p,fg))^{-1}$$, by 3.3.1(ii),

as $(pl-p)+fg = fq$ implies that $fq \le g$ and so we have

$$(pl-p)+(fg-g)+g = (fq-g)+g \text{ and } (pl-p)+(fg-g) = fq-g.$$

Also $\tau(fg-g,h)\theta_{pl-p} = \tau(pl-p,fg-g)\tau(fq-g,h)(\tau(pl-p,(fg-g)+h))^{-1}$

by the same argument as above and so

$$\tau(fg-g,h)\theta_{pl-p} = \tau(pl-p,fg-g)\tau(fq-g,h)(\tau(pl-p,p))^{-1}$$

Also $(b\theta_{fg-g})\theta_{pl-p} = \tau(pl-p,fg-g)\; b\theta_{(pl-p)+(fg-g)}(\tau(pl-p,fg-g))^{-1}$, by 3.2.1(3)

$$= \tau(pl-p,fg-g)\; b\theta_{fq-g}(\tau(pl-p,fg-g))^{-1}$$

Thus $v\theta_{pl-p} = \tau(pl-p,fg)(\tau(fq-g,g))^{-1}\, b\theta_{fq-g}\tau(fq-g,h)(\tau(pl-p,p))^{-1}$

and so $wx = (\tau(fq-f,e))^{-1}\, a\theta_{fq-f}\tau(fq-f,f)(\tau(fq-g,g))^{-1}\, b\theta_{fq-g} \times \tau(fq-g,h)(\tau(pl-1,1))^{-1}\, c\theta_{pl-1}\tau(pl-1,m)$.

A similar treatment of yz yields

$$yz = (\tau(fq-f,e))^{-1}\ a\theta_{fq-f}\tau(fq-f,f)(\tau(pl-h,g))^{-1}\ b\theta_{pl-h}\tau(pl-h,h)(\tau(pl-l,l))^{-1}$$
$$\times\ c\theta_{pl-l}\tau(pl-l,m).$$

For wx to be equal to yz and so to have the required result it is enough to show that pl-h = fq-g. We have from (3.3.1(iv)) that pl = (fq-q)+hl = (fq-q)+((hl-l)+l) = ((fq-q)+(hl-h))+h as the addition is associative. Hence pl ≤ h and pl-h = (fq-q)+(hl-h). However fq = (fq-q)+q = (fq-q)+((hl-h)+g) = ((fq-q)+(hl-h))+g, as the addition is associative. Thus fq ≤ g and fq-g = (fq-g)+(hl-h) and we have the required result.

We examine now the question of the 0-simplicity of $S(E, k, \Phi; A, \tau, \theta)$.

3.3.2 As in [1, Section 2.7] we make the following definition. In a semilattice E with zero, an element $f \in E^*$ is said to be primitive if $e \leq f$ implies $e = 0$ or $e = f$.

3.3.3 <u>Theorem</u>: (a) If E is a 0-uniform semilattice with no primitive idempotents then $S = S(E, k, \Phi; A, \tau, \theta)$ is 0-simple.

(b) If E is a 0-uniform semilattice with a primitive idempotent then $S = S(E, k, \Phi; A, \tau, \theta)$ is 0-simple if and only if A is simple.

<u>Proof</u>: (a) Let $(e,a,f), (g,b,h) \in S \setminus \{0\}$. Since E contains no primitive idempotents, there exists $l \in E^*$ with $l < e$. Let $v = (\tau(l-e,e))^{-1}\ a\theta_{l-e}\tau(l-e,f)$. Since $l-e \neq k$, $a\theta_{l-e} \in H_1$ and so $v \in H_1$. Consider $(g,b,l)(e,a,f)((l-e)+f,v^{-1},h) = (g,bv,(l-e)+f)((l-e)+f,v^{-1},h) = (g,bvv^{-1},h) = (g,b,h)$. Thus the semigroup S is 0-simple.

(b) Suppose that S is 0-simple. Let $a,b \in A$. Then, if $e \in E^*$, $(e,a,e), (e,b,e) \in S \setminus \{0\}$ so that there exist (f,c,g) and $(h,d,l) \in S \setminus \{0\}$

such that $(e,a,e) = (f,c,g)(e,b,e)(h,d,l)$. However, E has a primitive idempotent and so, as E is 0-uniform, every idempotent in E^* is primitive. Now $ge \leq e$ and so $ge = 0$ or $ge = e$. Since $(e,a,e) \neq 0$, $ge \neq 0$ and so $ge = e$. Thus $g \leq e$ so that $g = 0$ or $g = e$. Since $g \in E^*$, $g = e$. By a similar argument we have $h = e$. The outer components of the product $(f,c,e)(e,b,e)(e,d,l)$ are f and l. However since the product is equal to (e,a,e) we have $f = l = e$. Thus $(e,a,e) = (e,c,e)(e,b,e)(e,d,e) = (e,cbd,e)$ so that $a = cbd$ and A is simple.

Conversely assume that A is simple. Let (e,a,f), $(g,b,h) \in S\setminus\{0\}$. Since A is simple there exist $c,d \in A$ such that $a = cbd$. Consider the product $(e,c,g)(g,b,h)(h,d,f) = (e,cb,h)(h,d,f) = (e,cbd,f) = (e,a,f)$. Thus S is 0-simple.

In the following theorem we examine in detail the semigroup $S(E, k, \Phi; A, \tau, \theta)$.

3.3.4 Theorem: Let $S = S(E, k, \Phi; A, \tau, \theta)$. Then

1. (e,a,f) is an idempotent in $S\setminus\{0\} \iff e = f$ and $a = a^2$
2. S is regular $\iff$ A is regular
3. S is inverse $\iff$ A is inverse
4. $((e,a,f),(g,b,h)) \in \mathcal{L}_S \iff f = h$ and $(a,b) \in \mathcal{L}_A$

   $((e,a,f),(g,b,h)) \in \mathcal{R}_S \iff e = g$ and $(a,b) \in \mathcal{R}_A$

   $((e,a,f),(g,b,h)) \in \mathcal{H}_S \iff e = g$, $f = h$ and $(a,b) \in \mathcal{H}_A$

   $((e,a,f),(g,b,h)) \in \mathcal{D}_S \iff (a,b) \in \mathcal{D}_A$.
5. $\mathcal{H}$ is a congruence on S $\iff$ $\mathcal{H}$ is a congruence on A.
6. If A is an inverse semigroup then, for $a,b \in E_A$, $(e,a,e) \leq (f,b,f) \iff (e = f$ and $a \leq b)$ or $e < f$.
7. If A is an inverse semigroup then, for $a,b \in E_A$, $\{((e,a,e),(f,b,f)) \in \mathcal{H} \iff a = b\}$ holds $\iff$ A is a semilattice of groups.

Proof: 1. Let $(e,a,f) \varepsilon S\setminus\{0\}$ with $(e,a,f) = (e,a,f)(e,a,f)$. Then $ef \neq 0$ and $(e,a,f) = ((fe-f)+e,\ uv,\ (fe-e)+f)$ where $u = (\tau(fe-f,e))^{-1} a\theta_{fe-f}\tau(fe-f,f)$ and $v = (\tau(fe-e,e))^{-1}a\theta_{fe-e}\tau(fe-e,f)$. Thus $e = (fe-f)+e$ and $f = (fe-e)+f$ so that $k = fe-f = fe-e$ and so $f = fe = e$. From this we have $u = v = a$ and so $uv = a^2$. Thus $a = a^2$.

Conversely, consider $(e,a,e)(e,a,e)$, where $a = a^2$ in A and $e \varepsilon E^*$. We have $(e,a,e)(e,a,e) = (e,a^2,e) = (e,a,e)$ so that (e,a,e) is an idempotent in $S \setminus \{0\}$.

2. Assume that S is regular. Let $a \varepsilon A$. We have $(e,a,e) \varepsilon S$ and so, since S is regular, there exists $(f,b,g) \varepsilon S$ such that $(e,a,e) = (e,a,e)(f,b,g)(e,a,e)$. From this we see that $(e,a,e)(f,b,g) \neq 0$ and is an idempotent in S. Thus, by Theorem 3.3.4 (1), we know the form of $(e,a,e)(f,b,g)$. However, $(e,a,e)(f,b,g) = (ef,uv,(ef-f)+g)$ where $u = (\tau(ef-e,e))^{-1}a\theta_{ef-e}\tau(ef-e,e)$ and $v = (\tau(ef-f,f))^{-1}b\theta_{ef-f}\tau(ef-f,g)$. Hence $ef = (ef-f)+g$, and we have $(e,a,e)(f,b,g)(e,a,e) = (ef,uv,ef)(e,a,e) = (ef,(uv)w,ef)$ where $w = (\tau(ef-e,e))^{-1}a\ \theta_{ef-e}\tau(ef-e,e)$. However, $(e,a,e)(f,b,g)(e,a,e) = (e,a,e)$ so that $e = ef$, and $uvw = a$. If $ef = e$ then $u = a = w$ so that $ava = a$ and we have that A is regular.

Conversely, assume that A is regular. Let $(e,a,f) \varepsilon S\setminus\{0\}$. Since A is regular, there exists $b \varepsilon A$ such that $aba = a$. We consider $(e,a,f)(f,b,e)(e,a,f) = (e,ab,e)(e,a,f) = (e,aba,f) = (e,a,f)$ and we have that S is regular.

3. Assume that S is inverse. Then S is regular and so, by Theorem 3.3.4 (2), A is regular. To show that A is inverse we need only, by (1.1.4), show that any two idempotents in A commute. Let a and b be idempotents in A. Then, by Theorem 3.3.4 (1), (e,a,e) and (e,b,e) are idempotents in S. Thus, since S is inverse, $(e,a,e)(e,b,e) = (e,b,e)(e,a,e)$, and so $(e,ab,e) = (e,ba,e)$ and we have $ab = ba$.

Conversely, assume that A is inverse. Then, by Theorem 3.3.4 (2), S is regular. Let (e,a,e) and (f,b,f) be idempotents in S. By Theorem 3.3.4 (1), a and b are idempotents in A. Now, $(e,a,e)(f,b,f) = (ef,uv,ef)$ where $u = (\tau(ef-e,e))^{-1}a\theta_{ef-e}\tau(ef-e,e)$ and $v = (\tau(ef-f,f))^{-1}\times b\theta_{ef-f}\tau(ef-f,f)$. Also $(f,b,f)(e,a,e) = (ef,vu,ef)$. However $a\theta_{ef-e}$ and $b\theta_{ef-f}$ are idempotents in A, so that u and v are idempotents in A. Hence, since A is inverse, uv = vu and so (e,a,e)(f,b,f) = (f,b,f)(e,a,e) and S is inverse.

4. Let (e,a,f), (g,b,h) ε S with ((e,a,f),(g,b,h)) $\varepsilon\ \mathcal{L}$. Then there exist (l,c,m), (n,d,p) ε S such that (l,c,m)(e,a,f) = (g,b,h) and (n,d,p)(g,b,h) = (e,a,f). From these we have that (g,b,h) = $((me-m)+l,uv,(me-e)+f)$ where $u = (\tau(me-m,l))^{-1}\ c\theta_{me-m}\tau(me-m,m)$ and $v = (\tau(me-e,e))^{-1}\ a\theta_{me-e}\tau(me-e,f)$. Since h = (me-e)+f we have that $h \le f$. By considering (n,d,p)(g,b,h) = (e,a,f) we can show similarly that $f \le h$ and so f = h. Hence me-e = k and v = a so that b = ua. Similarly, by considering (n,d,p)(g,b,h) = (e,a,f) we have $u^1b = a$, for some $u^1 \in A$, and so (a,b) $\varepsilon\ \mathcal{L}$.

Conversely, let (e,a,f),(g,b,f) ε S with (a,b) $\varepsilon\ \mathcal{L}$. Then there exist c,d ε A such that ca = b and db = a. Considering (g,c,e)(e,a,f) = (g,ca,f) = (g,b,f) and (e,d,g)(g,b,f) = (e,db,f) = (e,a,f) we see that ((e,a,f),(g,b,f)) $\varepsilon\ \mathcal{L}$.

The result for $\mathcal{R}$ can be proved in a similar manner and the result for $\mathcal{H}$ then follows immediately.

Let (e,a,f),(g,b,h) ε S and suppose that ((e,a,f),(g,b,h)) $\varepsilon\ \mathcal{D}$ Then there exists (m,c,n) ε S such that ((e,a,f),(m,c,n)) $\varepsilon\ \mathcal{R}$ and ((m,c,n),(g,b,h)) $\varepsilon\ \mathcal{L}$. From the above results we have (a,c) $\varepsilon\ \mathcal{R}$ and (c,b) $\varepsilon\ \mathcal{L}$ so that (a,b) $\varepsilon\ \mathcal{D}$.

Conversely, let $(e,a,f),(g,b,h) \varepsilon S$ with $(a,b) \varepsilon \mathcal{D}$. Then there exists $c \varepsilon A$ such that $(a,c) \varepsilon \mathcal{R}$ and $(c,b) \varepsilon \mathcal{L}$. Hence, from the above results, $((e,a,f),(e,c,h)) \varepsilon \mathcal{R}$ and $((e,c,h),(g,b,h)) \varepsilon \mathcal{L}$ so that $((e,a,f),(g,b,h)) \varepsilon \mathcal{D}$.

5. Assume that $\mathcal{H}$ is a congruence on S. Let $a,b \varepsilon A$ with $(a,b) \varepsilon \mathcal{H}$ and let $c,d \varepsilon A$. By Theorem 3.3.4(4), we have that $((e,a,e),(e,b,e)) \varepsilon \mathcal{H}$ and, as $\mathcal{H}$ is a congruence on S, $((e,c,e)(e,a,e),(e,c,e)(e,b,e)) \varepsilon \mathcal{H}$. However $(e,c,e)(e,a,e) = (e,ca,e)$ and $(e,c,e)(e,b,e) = (e,cb,e)$ so that, by Theorem 3.3.4 (4), we have $(ca,cb) \varepsilon \mathcal{H}$. Similarly, by considering $((e,a,e)(e,d,e),(e,b,e)(e,d,e))$ we have $(ad,bd) \varepsilon \mathcal{H}$ and so $\mathcal{H}$ is a congruence on A.

Conversely, assume that $\mathcal{H}$ is a congruence on A. Let (e,a,f), $(e,b,f) \varepsilon S$ with $((e,a,f),(e,b,f)) \varepsilon \mathcal{H}$. Then, by Theorem 3.3.4 (4), $(a,b) \varepsilon \mathcal{H}$. Let $(g,c,h),(l,d,m) \varepsilon S$. We consider $(g,c,h)(e,a,f)$ and $(g,c,h)(e,b,f)$. If $he = 0$ then both products are zero and so are $\mathcal{H}$ equivalent. If $he \neq 0$ then $(g,c,h)(e,a,f) = ((he-h)+g, uv, (he-e)+f)$ where $u = (\tau(he-h,g))^{-1} c\theta_{he-h}\tau(he-h,h)$ and $v = (\tau(he-e,e))^{-1}a\theta_{he-e} \times \tau(he-e,f)$. Also $(g,c,h)(e,b,f) = ((he-h)+g,uw,(he-e)+f)$ where $w = (\tau(he-e,e))^{-1}b\theta_{he-e}\tau(he-e,f)$. Since $(a,b) \varepsilon \mathcal{H}$ we also have $(a\theta_{he-e},b\theta_{he-e}) \varepsilon \mathcal{H}$ as θ_{he-e} is an endomorphism of A. Hence, as $\mathcal{H}$ is a congruence on A, $(v,w) \varepsilon \mathcal{H}$ and so $(uv,uw) \varepsilon \mathcal{H}$. Applying Theorem 3.3.4(4), we now have $((g,c,h)(e,a,f), (g,c,h)(e,b,f)) \varepsilon \mathcal{H}$. It can be shown similarly that $((e,a,f)(l,d,m), (e,b,f)(l,d,m)) \varepsilon \mathcal{H}$ and so $\mathcal{H}$ is a congruence on S.

6. Let (e,a,e) and (f,b,f) be idempotents in S. Since A is inverse, S is inverse, by Theorem 3.3.4 (3), and so the set of idempotents of S forms a semilattice. Assume that $(e,a,e) \leq (f,b,f)$. Then $(e,a,e) = (e,a,e)(f,b,f) = (ef,uv,ef)$ where $u = (\tau(ef-e,e))^{-1}a\theta_{ef-e}\tau(ef-e,e)$ and $v = (\tau(ef-f,f))^{-1}b\theta_{ef-f}\tau(ef-f,f)$.

Hence $e = ef$ and so $e \leq f$, and also $a = uv$. If $e = f$ then $u = a$ and $v = b$ so that $a = ab$ and $a \leq b$. Thus $e < f$ or $e = f$ and $a \leq b$.

Conversely assume that (e,a,e), (f,b,f) are idempotents in S with $e < f$. Then $(e,a,e)(f,b,f) = (e,ax,e)$ where $x = (\tau(e-f,f))^{-1} \chi$ $b\theta_{e-f}\tau(e-f,f)$. As $e < f$, $e-f \neq k$ and so $b\theta_{e-f} = 1$ and $x = 1$. Thus $(e,a,e)(f,b,f) = (e,a,e)$. If $e = f$ and $a \leq b$ then $(e,a,e)(f,b,f) = (e,ab,e) = (e,a,e)$. Hence in either case $(e,a,e) \leq (f,b,f)$.

7. Suppose that S is such that, when $a,b \in E_A$, $((e,a,e), (f,b,f)) \in \mathcal{D} \Leftrightarrow a = b$. Let $c \in A$, Then $(cc^{-1}, c^{-1}c) \in \mathcal{D}$ in A and so, by Theorem 3.3.4 (4), $((e,cc^{-1},e), (f,c^{-1}c,f)) \in \mathcal{D}$, where $e,f \in E^*$. Hence $cc^{-1} = c^{-1}c$ and so by (1.2.12), since A is inverse, A is a semilattice of groups.

Conversely let A be a semilattice of groups. Clearly if $a \in E_A$, by Theorem 3.3.4 (4), $((e,a,e),(f,a,f)) \in \mathcal{D}$ for $e,f \in E^*$. If $a,b \in E_A$ and $((e,a,e), (f,b,f)) \in \mathcal{D}$, where $e,f \in E^*$, then, by Theorem 3.3.4 (4), $(a,b) \in \mathcal{D}$. Thus there exists $c \in A$ such that $cc^{-1} = a$ and $c^{-1}c = b$. However A is a semilattice of groups and is inverse so that by (1.2.12) $cc^{-1} = c^{-1}c$ and so $a = b$.

3.3.5 If we now examine more closely those semigroups of the form $S(E, k, \Phi; A, \tau, \theta)$ where A is a centric inverse monoid and E has a primitive idempotent then by Theorem 3.3.3(b) we require that A is simple so that $S(E, k, \Phi; A, \tau, \theta)$ is a 0-simple inverse semigroup. However, by [1, Theorem 4.5] A, in this case, is itself completely simple and so by [1, Section 2.7] has a primitive idempotent. Applying Theorem 3.3.4 (6), we readily have that $S(E, k, \Phi; A, \tau, \theta)$ has a primitive idempotent so that S itself is completely 0-simple. However, in [1, Theorem 3.5] the Rees Theorem determining the structure of completely 0-simple semigroups is stated.

3.4 Factorisation of a semilattice E_S compatible with the $\mathcal{D}$-structure of S

3.4.1 Let S be an inverse semigroup with zero, and semilattice E. Then E is said to admit a factorisation compatible with the $\mathcal{D}$-structure of S if:- (i) E^* is order isomorphic to $F^* \times Y$ where F is a semilattice with zero and Y is a semilattice with identity, and where $(f,\alpha) \leq (g,\beta)$ in $F^* \times Y \iff (f = g$ and $\alpha \leq \beta)$ or $f < g$.

(ii) if $e,f \in E^*$ and, under the order isomorphism of (i), $e \to (g,\alpha)$ in $F^* \times Y$ and $f \to (h,\beta)$ in $F^* \to Y$, then $(e,f) \in \mathcal{D}_S \iff \alpha = \beta$ in Y.

3.4.2 This is a formalisation of the situation described in Theorem 3.3.4 (1), (6) and (7). We thus have that if A is an inverse monoid which is a semilattice of groups then $S = S(E, k, \mathfrak{I}; A, \tau, \theta)$ is such that E_S admits a factorisation compatible with the $\mathcal{D}$-structure of S. In fact $E_S \setminus \{0\} \cong E^* \times E_A$.

3.4.3 Theorem: Let S be a 0-simple inverse semigroup with semilattice E which admits a factorisation compatible with the $\mathcal{D}$-structure of S as in 3.4.1. Let E^* be order isomorphic to $F^* \times Y$. Then F is a 0-uniform semilattice.

Proof: Let 1 denote the identity of Y and take $E^* = F^* \times Y$. We show firstly that $\bar{F} = \{(e,1): e \in F^*\} \cup \{0\}$ is a semilattice isomorphic to F. Since $\bar{F} \subset E$, $\bar{F}$ is a partially ordered set. Let $(e,1),(f,1) \in \bar{F}$. Then, if $ef \neq 0$, $(ef,1) \in \bar{F}$. However $(ef,1) \leq (e,1),(f,1)$ so that $(ef,1) \leq (e,1)(f,1)$ this latter product being defined in E. Let $(e,1)(f,1) = (x,\alpha)$, say. Then $(x,\alpha) \leq (e,1),(f,1)$ and so $x \leq e,f$ and $x \leq ef$. Thus $(x,\alpha) \leq (ef,1)$ so that $(e,1)(f,1) = (ef,1)$. If $ef = 0$ then $(e,1)(f,1) = 0$ also. Hence $\bar{F}$ is a semilattice. The mapping $(e,1) \to e$ and $0 \to 0$ is a semilattice isomorphism from $\bar{F}$ onto F.

Let $e,f \in F^*$. Then $((e,1),(f,1)) \in \mathcal{D}$ and so $D_{(e,1)} = D_{(f,1)}$. Let $B = D_{(e,1)} \cup \{0\}$ for $e \in F^*$. Because of the factorisation of E compatible with the $\mathcal{D}$-structure of S, $D_{(e,1)} = \bigcup_{f \in F^*} R_{(f,1)} = \bigcup_{f \in F^*} L_{(f,1)}$. Hence $B = (\bigcup_{f \in F^*} R_{(f,1)}) \cup \{0\} = (\bigcup_{f \in F^*} L_{(f,1)}) \cup \{0\}$. Let $x,y \in B\backslash\{0\}$. Then there exist $f,g \in F^*$ such that $(x,(f,1)) \in \mathcal{L}$ and $(y,(g,1)) \in \mathcal{R}$. Thus $Sx = S(f,1)$ and $yS = (g,1)S$. From this we have $Sxy = S(f,1)y$ and $(f,1)yS = (f,1)(g,1)S$. From the first part of the theorem we thus have $xy = 0$ or $(xy, (fg,1)) \in \mathcal{D}$, $fg \neq 0$. Clearly in either case $xy \in B$ and so B is a subsemigroup of S. If $x \in B\backslash\{0\}$ then we have immediately that $x^{-1} \in B$ and so B is an inverse subsemigroup of S. The semilattice of B is $\bar{F}$. Let $(e,1),(f,1) \in \bar{F}$. Then, as $((e,1),(f,1)) \in \mathcal{D}_S$ there exists $x \in S$ such that $(e,1) = xx^{-1}$ and $(f,1) = x^{-1}x$, from [6, Lemma 1.1]. However $x \in R_{(e,1)}$ and so $x \in B$. Thus $((e,1),(f,1)) \in \mathcal{D}_B$. Hence by [6, Lemma 1.1] B is 0-bisimple and so, by [10, Theorem 1.2], $\bar{F}$ is 0-uniform and F is also 0-uniform.

3.5 The structure of a type of 0-simple inverse semigroup

3.5.1 In this section we set out to show that a certain type of 0-simple inverse semigroup whose semilattice admits a factorisation compatible with the $\mathcal{D}$-structure of the semigroup is of the form described in sections 3.2 and 3.3. We break the proof into two main sections: the first consists of showing that $\mathcal{H}$ is a congruence on a semigroup of the type being considered and then examining $S/\mathcal{H}$; the second consists of examining the semigroup itself.

3.5.2 Before stating the first theorem we recall from (1.3.1) that μ denotes the maximal idempotent separating congruence on an inverse semigroup S. Also, from [6, Lemma 3.1] and [6, Lemma 1.2], if S is an inverse semigroup, then $S/\mu \cong S\theta$ where the homomorphism $\theta: S \to T_{E_S}$ is such that $a\theta = \theta_a : E_S aa^{-1} \to E_S a^{-1}a$, with $e\theta_a = a^{-1}ea$ for all $e \in E_S aa^{-1}$, is an isomorphism.

3.5.3 <u>Theorem</u>: Let S be a 0-simple inverse semigroup whose semilattice E admits a factorisation compatible with the $\mathcal{D}$-structure of S. Let this factorisation be given by $E^* = F^* \times Y$ where F contains a non zero principal ideal whose group of order automorphisms is trivial. Then $\mathcal{H}$ is a congruence on S.

<u>Proof</u>: We have by Theorem 3.4.3 that F is a 0-uniform semilattice. Hence every non zero principal ideal of F has a trivial group of order automorphisms. From this we can deduce that, if $x,y \in F^*$, there exists a unique isomorphism from Fx onto Fy. As in (1.3.10) and (1.3.11) denote this mapping by $\xi_{x,y}$ and so we have $T_F = \{\xi_{x,y} : x,y \in F^*\} \cup \{0\}$ with multiplication as described in (1.3.11).

We wish to show that $\mathcal{H}$ is a congruence on S. By (1.3.1) we have $\mu \subseteq \mathcal{H}$ so that to prove $\mathcal{H}$ is a congruence we need only show that $\mathcal{H} \subseteq \mu$. Let $a,b \in S\backslash\{0\}$ with $(a,b) \in \mathcal{H}$. Then $aa^{-1} = bb^{-1} = (x,\alpha)$, (say), and $a^{-1}a = b^{-1}b = (y,\alpha)$, (say): the Y-components of the idempotents are equal since $(aa^{-1}, a^{-1}a) \in \mathcal{D}$. Thus $\theta_a : E(x,\alpha) \to E(y,\alpha)$ and $\theta_b : E(x,\alpha) \to E(y,\alpha)$.

The following lemma facilitates the completion of the proof.

3.5.4 <u>Lemma</u>: Let S be a 0-simple inverse semigroup as described in Theorem 3.5.3. For $a \in S\backslash\{0\}$ with $\theta_a : E(x,\alpha) \to E(y,\alpha)$, $(Z,\eta)\theta_a = (Z\xi_{x,y}, \eta)$ for all $(Z,\eta) \in E(x,\alpha)$.

<u>Proof</u>: We note that if $p \in Eaa^{-1}$ then $(p,pa) \in \mathcal{R}$ as $pa(pa)^{-1} = paa^{-1}p^{-1} = pp^{-1}$. Also $(pa, a^{-1}pa) \in \mathcal{L}$ as $(a^{-1}pa)^{-1}(a^{-1}pa) = a^{-1}p^{-1}aa^{-1}pa = a^{-1}p^{-1}pa = (pa)^{-1}pa$. Thus $(p, a^{-1}pa) = (p, p\theta_a) \in \mathcal{D}$.

Hence if $(Z,\eta) \in E(x,\alpha)$ then $((Z,\eta), (Z,\eta)\theta_a) \in \mathcal{D}$ and so, from the $\mathcal{D}$-compatibility of the factorisation of E, we have $(Z,\eta)\theta_a = (w,\eta)$ (say). We note that $E(x,\alpha) = \bigcup_{\substack{\eta\in Y\\ \eta\leq\alpha}} \{(p,\eta):p \not> x\} \cup \bigcup_{\substack{\delta\in Y\\ \delta\not\leq\alpha}} \{(p,\delta): p < x\}$.

For $\eta\in Y$ define α_η on Fx as follows:-

$x\alpha_\eta = y$, $0\alpha_\eta = 0$ and, if $p < x$, $(p\alpha_\eta,\eta) = (p,\eta)\theta_a$.

It is easily checked that α_η:Fx → Fy is an isomorphism so that $\alpha_\eta = \xi_{x,y}$ and we have the result.

Returning to the theorem we have that, for all $(p,\eta) \in E(x,\alpha)$ $(p,\eta)\theta_a = (p\xi_{x,y},\eta) = (p,\eta)\theta_b$ and so $\theta_a = \theta_b$. From [6, Lemma 3.1] we now have $(a,b)\in\mu$ and the result follows.

3.5.6 <u>Theorem:</u> Let S be a 0-simple inverse semigroup whose semilattice E admits a factorisation compatible with the $\mathcal{D}$-structure of S. Let this factorisation be given by $E^* = F^* \times Y$ where F contains a non zero principal ideal whose group of order automorphisms is trivial. Then $S/\mathcal{H}$ is of the form $S(F, k, \Phi; Y, \tau, \psi)$.

<u>Proof:</u> First, by Theorem 3.4.3, F is 0-uniform. As noted in Theorem 3.5.3 we have $T_F = \{\xi_{x,y}:x,y \in F^*\}\cup\{0\}$ with multiplication as in (1.3.11). Fix $k \in F^*$ and define $\Phi(f)$ to be $\xi_{k,f}$ for all $f \in F^*$. Clearly Φ is an addition on F with identity k. We consider whether the addition is associative. Let $e,f \in F^+$, where $F^+ = \{x \in F^*:x\leq k\}$, and $g \in F^*$, then $(e+f)+g = (e\xi_{k,f})\xi_{k,g} = e(\xi_{k,f})\xi_{k,g} = (e\xi_{k,f\xi_{k,g}}) = e\xi_{k,f+g} = e+(f+g)$, which shows that the addition is associative.

By Theorem 3.5.3, $\mathcal{H}$ is a congruence on S and so by [6, Lemma 3.1] $S/\mathcal{H} = S/\mu \simeq S\theta$ where θ is as defined in (3.5.2). Define a mapping $\phi:S\theta \to F^* \times Y \times F^*\cup\{0\}$ as follows:-

$$0\phi = 0$$

$$(\theta_a)\phi = (e,\alpha,f) \text{ where } aa^{-1} = (e,\alpha) \text{ and } a^{-1}a = (f,\alpha), \text{ for } \alpha \neq 0.$$

This mapping is well-defined; for if $\theta_a = \theta_b$ then, by [6, Lemma 3.1], $(a,b) \in \mathcal{H}$ and $aa^{-1} = bb^{-1}$ and $a^{-1}a = b^{-1}b$. Also if $\theta_a, \theta_b \in S\theta$ with $(\theta_a)\phi = (\theta_b)\phi$ then $aa^{-1} = bb^{-1}$ and $a^{-1}a = b^{-1}b$ so that $(a,b) \in \mathcal{H}$ and $(a,b) \in \mu$. Hence, by [6, Lemma 3.1], $\theta_a = \theta_b$. If $(e,\alpha,f) \in F^* \times Y \times F^*$ then $((e,\alpha),(f,\alpha)) \in \mathcal{D}$ in S by the $\mathcal{D}$-compatibility of the factorisation of E. Hence there exists $a \in S$ such that $((e,\alpha),a) \in \mathcal{R}$ and $((f,\alpha),a) \in \mathcal{L}$. From this we have $aa^{-1} = (e,\alpha)$ and $a^{-1}a = (f,\alpha)$ and $(\theta_a)\phi = (e,\alpha,f)$. Combining all this information we have that ϕ is a bijection on $S\theta$.

Define τ, ψ by $\tau(f,g) = 1$ for all $(f,g) \in F^+ \times F^*$ and $\alpha\psi_e = 1$ for all $\alpha \in Y$, $e \in E^*$, $e \neq k$. To complete the proof we must examine $(\theta_a\theta_b)\phi$ and $(\theta_a)\phi(\theta_b)\phi$ for $\theta_a, \theta_b \in S\theta$, where $a, b \neq 0$. Let $(\theta_a)\phi = (e,\alpha,f)$ and $(\theta_b)\phi = (g,\beta,h)$. If $fg = 0$ then in $S(F, k, \Phi; Y, \tau, \psi)$ we have $(\theta_a)\phi(\theta_b)\phi = 0$. Also $(f,\alpha)(g,\beta) = 0 = a^{-1}abb^{-1}$; hence $ab = 0$ and $(\theta_a\theta_b)\phi = (\theta_{ab})\phi = 0$ also. If $fg \neq 0$ then, in $S(F, k, \Phi; Y, \tau, \psi)$, $(\theta_a)\phi(\theta_b)\phi = (e,\alpha,f)(g,\beta,h) = ((fg-f)+e, uv, (fg-g)+h)$ where $u = (\tau(fg-f,e))^{-1}\alpha\psi_{fg-f}\,\tau(fg-f,f) = \alpha\psi_{fg-f}$ and $v = (\tau(fg-g,g))^{-1}\beta\psi_{fg-g}\tau(fg-g,h) = \beta\psi_{fg-g}$, this simplification being possible as $H_1 = \{1\}$ and so $\tau(f,g) = 1$ for all $f \in F^+$, $g \in F^*$, where 1 is the identity element of Y. There are four possible situations:-

(i) $f = g$ and so $uv = \alpha\beta$

(ii) $f < g$ and so $uv = \alpha 1 = \alpha$

(iii) $f > g$ and so $uv = 1\beta = \beta$

(iv) f and g are incomparable and so $uv = 1$.

On the other hand, consider $(\theta_a\theta_b)\phi$. We have $\Delta(\theta_a\theta_b) = (E(f,\alpha)(g,\beta))\theta_a^{-1} = (E(f,\alpha)(g,\beta))\theta_{a^{-1}} = E((f,\alpha)(g,\beta))\theta_{a^{-1}}$ by [6, Lemma 2.1]. Similarly $\nabla(\theta_a\theta_b) = E((f,\alpha)(g,\beta))\theta_b$. In case (i) $(f,\alpha)(g,\beta) = (f,\alpha\beta)$; in case (ii) $(f,\alpha)(g,\beta) = (f,\alpha)$; in case (iii) $(f,\alpha)(g,\beta) = (g,\beta)$; in case (iv) $(f,\alpha)(g,\beta) = (fg,1)$. Thus in each case $\Delta(\theta_a\theta_b) =$

$E(fg,\gamma)\theta_{a^{-1}}$ and $\nabla(\theta_a\theta_b) = E(fg,\gamma)\theta_b$ where $\gamma = \alpha\beta, \alpha, \beta$ or 1 as appropriate. Applying Lemma 3.5.4 we have that $\Delta(\theta_a\theta_b) = E(fg\xi_{f,e},\gamma)$ and $\nabla(\theta_a\theta_b) = E(fg\xi_{g,h},\gamma)$. However $fg\xi_{f,e} = fg\xi_{f,k}\xi_{k,e} = fg\xi_{k,f}^{-1}\xi_{k,e} = (fg-f)+e$ and $fg\xi_{g,h} = fg\xi_{g,k}\xi_{k,h} = fg\xi_{k,g}^{-1}\xi_{k,h} = (fg-g)+h$, and we have $(\theta_a\theta_b)\phi = ((fg-f)+e,\ \gamma,\ (fg-g)+h)$ where $\gamma = \alpha\beta$ in case (i), $\gamma = \alpha$ in case (ii), $\gamma = \beta$ in case (iii) and $\gamma = 1$ in case (iv). Hence $(\theta_a\theta_b)\phi = (\theta_a)\phi(\theta_b)\phi$ and we have that ϕ is an isomorphism.

We now make the step from obtaining the structure of $S/\mathcal{H}$ to obtaining the structure of S.

3.5.7 Theorem Let S be a 0-simple inverse semigroup whose semilattice E admits a factorisation compatible with the $\mathcal{D}$-structure of S. Let this factorisation be given by $E^* = F^* \times Y$ where F has a non zero principal ideal whose group of order automorphisms is trivial. Then $S \cong S(F, k, \Phi\ ;\ A, \tau, \psi)$ where A is a centric inverse monoid with semilattice Y.

Proof: By Theorem 3.4.3, F is 0-uniform. Take $k \in F^*$ to be as in Theorem 3.5.6 and define the addition Φ as in Theorem 3.5.6. Let 1 denote the identity element of Y.

For each $\alpha \in Y$ let $G_\alpha = H_{(k,\alpha)}$ and let $A = \bigcup_{\alpha \in Y} G_\alpha$. Then each G_α is a group by (1.2.11). Let $x,y \in A$ with $x \in G_\alpha$ and $y \in G_\beta$. Then $(x,(k,\alpha)) \in \mathcal{H}$ and $(y,(k,\beta)) \in \mathcal{H}$. However, by Theorem 3.5.3, $\mathcal{H}$ is a congruence on S and so $(xy,(k,\alpha)y) \in \mathcal{H}$ and $((k,\alpha)y,(k,\alpha)(k,\beta)) \in \mathcal{H}$. Hence $(xy,(k,\alpha)(k,\beta)) \in \mathcal{H}$. But $(k,\alpha)(k,\beta) = (k,\alpha\beta)$ in E and so $(xy,(k,\alpha\beta)) \in \mathcal{H}$ and $xy \in G_{\alpha\beta}$. Thus $xy \in A$ and A is a subsemigroup of S. Moreover, since $G_\alpha G_\beta \subseteq G_{\alpha\beta}$, A is a semilattice of groups. The idempotent $(k,1) \in A$ and is an identity for the semigroup A so that A is a centric inverse monoid with semilattice $\{(k,\alpha):\alpha \in Y\} \cong Y$.

From Theorem 3.5.6 we have that there exists an isomorphism $\phi: S/\mathcal{H} \to F^* \times Y \times F^* \cup \{0\}$ whereby $0\phi = 0$ and $(H_s)\phi = (e,\alpha,f)$ where $ss^{-1} = (e,\alpha)$ and $s^{-1}s = (f,\alpha)$, the multiplication in $F^* \times Y \times F^* \cup \{0\}$ being as described in Theorem 3.5.6.

We select a transversal of the $\mathcal{H}$-classes of S contained in $R_{(k,1)}$ as follows:-

Let u_e be the representative of H_a where $(H_a)\phi = (k,1,e)$; we stipulate that $u_k = (k,1)$.

We now use this transversal to obtain a unique form for the elements of S and establish this in the following lemma.

3.5.8 <u>Lemma</u>. If $x \in S\setminus\{0\}$ there is a unique representation of x in the form $u_e^{-1}g_\alpha u_f$ where $(H_x)\phi = (e,\alpha,f)$ and $g_\alpha \in G_\alpha$.

<u>Proof</u>: Let $x \in S\setminus\{0\}$ with $(H_x)\phi = (e,\alpha,f)$. Then $(e,\alpha,f) = (e,1,k)(k,\alpha,k)(k,1,f)$ and we have $(H_x)\phi = ((H_{u_e})\phi)^{-1}(H_{(k,\alpha)})\phi(H_{u_f})\phi = (H_{u_e^{-1}(k,\alpha)u_f})\phi$ as ϕ is an isomorphism and $\mathcal{H}$, by Theorem 3.5.3, is a congruence on S. Thus $H_x = H_{u_e^{-1}(k,\alpha)u_f}$ and $(x, u_e^{-1}(k,\alpha)u_f) \in \mathcal{H}$. Again using the fact that $\mathcal{H}$ is a congruence on S, we have $(u_e x u_f^{-1}, u_e u_e^{-1}(k,\alpha)u_f u_f^{-1}) \in \mathcal{H}$. However, $u_e u_e^{-1} = (k,1) = u_f u_f^{-1}$ as $u_e, u_f \in R_{(k,1)}$. Thus $(u_e x u_f^{-1}, (k,\alpha)) \in \mathcal{H}$, and so $u_e x u_f^{-1} \in H_{(k,\alpha)} = G_\alpha$. Let $u_e x u_f^{-1} = g_\alpha$ (say) $\in G_\alpha$. Then $u_e^{-1}u_e x u_f^{-1}u_f = u_e^{-1}g_\alpha u_f$. However $u_e^{-1}u_e = (e,\alpha)$ and $u_f^{-1}u_f = (f,\alpha)$, while $xx^{-1} = (e,\alpha)$ and $x^{-1}x = (f,\alpha)$, so that $u_e^{-1}u_e x u_f^{-1}u_f = x$ and we have $x = u_e^{-1}g_\alpha u_f$. Thus there is a representation of x in the required form and it remains to prove that it is unique.

Suppose that $x \in S\setminus\{0\}$ and $x = u_e^{-1}g_\alpha u_f$ and $x = u_p^{-1}h_\beta u_q$. Then $(H_x)\phi = (e,1,k)(k,\alpha,k)(k,1,f) = (e,\alpha,f)$ and $(H_x)\phi = (p,1,k)(k,\beta,k) \times (k,1,q) = (p,\beta,q)$ so that $e = p$, $f = q$ and $\alpha = \beta$. Hence $x = u_e^{-1}g_\alpha u_f = u_e^{-1}h_\alpha u_f$ so that $u_e x u_f^{-1} = u_e u_e^{-1} g_\alpha u_f u_f^{-1}$ and $u_e x u_f^{-1} = u_e u_e^{-1} h_\alpha u_f u_f^{-1}$. Since $u_e u_e^{-1} = u_f u_f^{-1} = (k,1)$ we now have $u_e x u_f^{-1} = g_\alpha$ and $u_e x u_f^{-1} = h_\alpha$ so that $g_\alpha = h_\alpha$ and this representation is indeed unique.

Returning now to the theorem, this representation leads us to define a mapping $\sigma: S \to F^* \times A \times F^* \cup \{0\}$ as follows:-

$0\sigma = 0$ and $s\sigma = (e, g_\alpha, f)$ where $(H_s)\phi = (e, \alpha, f)$ and $s = u_e^{-1} g_\alpha u_f$.

Clearly σ is a well-defined mapping. If $x, y \in S$ with $x\sigma = y\sigma = (e, g_\alpha, f)$ (say), then $x = u_e^{-1} g_\alpha u_f = y$ so that σ is injective. Also, if $(e, g_\alpha, f) \in F^* \times A \times F^*$ let $x = u_e^{-1} g_\alpha u_f$. Then $(H_x)\phi = (e, \alpha, f)$ and $x\sigma = (e, g_\alpha, f)$ so that σ is surjective. To show that σ is a homomorphism we need to examine the multiplication and to simplify this we introduce the following lemmas.

3.5.9 <u>Lemma</u>: If $e \in F^+$ and $f \in F^*$ then there exists a unique element $g(e,f)$ in G_1 such that $u_e u_f = g(e,f) u_{e+f}$.

<u>Proof</u>: Since ϕ is an isomorphism on $S/\mathcal{H}$ and $\mathcal{H}$ is a congruence on S we have $(H_{u_e u_f})\phi = (H_{u_e})\phi (H_{u_f})\phi = (k,1,e)(k,1,f) = (k,1,e+f)$. Thus $(H_{u_e u_f})\phi = (k,1,e+f)$ and, from Lemma 3.5.8, we have that there exists a unique element g_1 in G_1 such that $u_e u_f = u_k^{-1} g_1 u_{e+f} = g_1 u_{e+f}$.

3.5.10 From here onwards we shall, for all $e \in F^+$ and $f \in F^*$, denote by $\tau(e,f)$ the unique element in G_1 such that $u_e u_f = \tau(e,f) u_{e+f}$.

3.5.11 <u>Lemma</u>: For all $e \in F^+$ and $f \in F^*$ we have $\tau(e,k) = (k,1) = \tau(k,f)$.

<u>Proof</u>: From (3.5.10), $\tau(e,k) u_{e+k} = u_e u_k$ and so $\tau(e,k) u_e = u_e$. Hence $\tau(e,k) u_e u_e^{-1} = u_e u_e^{-1}$. However $u_e u_e^{-1} = (k,1)$ and we have $\tau(e,k) = (k,1)$. From (3.5.10), $\tau(k,f) u_{k+f} = u_k u_f$ and so $\tau(k,f) u_f = u_f$. Hence $\tau(k,f) u_f u_f^{-1} = u_f u_f^{-1}$. However $u_f u_f^{-1} = (k,1)$ and we have $\tau(k,f) = (k,1)$.

3.5.12 <u>Lemma</u>: For all $e,f \in F^*$, $u_e u_f^{-1} = u_{ef-e}^{-1} \tau(ef-e,e) (\tau(ef-f,f))^{-1} \times u_{ef-f}$.

Proof: We have $(H_{u_e u_f^{-1}})\phi = (H_{u_e})\phi((H_{u_f})\phi)^{-1}$ since $\mathcal{H}$ is a congruence on S and ϕ is a homomorphism. Thus we have

$(H_{u_e u_f^{-1}})\phi = (k,1,e)(f,1,k) = (ef-e,1,ef-f)$ and, by Lemma 3.5.8, $u_e u_f^{-1} = u_{ef-e}^{-1}\ g_1\ u_{ef-f}$ where $g_1 \in G$. However, $u_{ef-e} u_e u_f^{-1} u_{ef-f}^{-1}$ $= u_{ef-e} u_{ef-e}^{-1}\ g_1\ u_{ef-f} u_{ef-f}^{-1}$ and as $u_{ef-e} u_{ef-e}^{-1} = (k,1) = u_{ef-f} \times u_{ef-f}^{-1}$ we have $g_1 = u_{ef-e} u_e (u_{ef-f} u_f)^{-1}$. From (3.5.10) we have $u_{ef-e} u_e = \tau(ef-e,e) u_{ef}$ and $u_{ef-f} u_f = \tau(ef-f,f) u_{ef}$. Hence $g_1 = \tau(ef-e,e) u_{ef} u_{ef}^{-1} (\tau(ef-f,f))^{-1}$. However $u_{ef} u_{ef}^{-1} = (k,1)$ and so $g_1 = \tau(ef-e,e)(\tau(ef-f,f))^{-1}$ from which we obtain the required result.

3.5.13 Lemma: Let $e \in F^+ \setminus \{k\}$ and $a \in A$. Then there exists a unique element $g \in G_1$ such that $u_e a = g u_e$.

Proof: Let $a \in A$ with $a \in G_\alpha$ (say). Then $(H_{u_e a})\phi = (H_{u_e})\phi(H_a)\phi$ $= (k,1,e)(k,\alpha,k)$ as $\mathcal{H}$ is a congruence on S and ϕ is a homomorphism. Hence $(H_{u_e a})\phi = (k,1,e)$ and so by Lemma 3.5.8 we have $u_e a = u_k^{-1}\ g_1 u_e$ for a unique element $g_1 \in G_1$. However $u_k = (k,1)$ and so $u_e a = g_1 u_e$ for a unique element $g_1 \in G_1$.

3.5.14 Define a mapping $\psi_e : A \to G_1$, for all $e \in F^+ \setminus \{k\}$, as follows:- $u_e a = (a\psi_e) u_e$. By Lemma 3.5.13 this mapping is well-defined for each $e \in F^+ \setminus \{k\}$. Also, as $u_e(ab) = (u_e a)b = (a\psi_e) u_e b =$ $(a\psi_e)(b\psi_e) u_e$ we have $(ab)\psi_e u_e = (a\psi_e)(b\psi_e) u_e$ so that, on post-multiplying by u_e^{-1}, we have $(ab)\psi_e = (a\psi_e)(b\psi_e)$. Hence for each $e \in F^+ \setminus \{k\}$ we have that ψ_e is a homomorphism from A into G_1. Define ψ_k to be the identity mapping on A.

3.5.15 Lemma: If $e,f \in F^+$ and $g \in F^*$ then $\tau(e,f)\tau(e+f,g) =$ $(\tau(f,g))\psi_e\ \tau(e,f+g)$.

Proof: If $e = k$ the result is immediate once we note that $\tau(e,f) = \tau(e,f+g) = (k,1)$, by Lemma 3.5.11.

If $e \neq k$ then by (3.5.10) and (3.5.14) we have $u_e(u_f u_g)$ $= u_e\ \tau(f,g)u_{f+g} = (\tau(f,g))\psi_e u_e u_{f+g} = \tau(f,g)\psi_e\ \tau(e,f+g)u_{e+(f+g)}$. On the other hand, by (3.5.10), $(u_e u_f)u_g = \tau(e,f)u_{e+f}u_g = \tau(e,f) \times \tau(e+f,g)u_{(e+f)+g}$. As the addition is associative $u_{(e+f)+g} = u_{e+(f+g)}$ and so $\tau(f,g)\psi_e\ \tau(e,f+g)u_{e+f+g} = \tau(e,f)\tau(e+f,g)u_{e+f+g}$. On post multiplying by u^{-1}_{e+f+g} and noting that $u_{e+f+g}\ u^{-1}_{e+f+g} = (k,1)$ we have the result.

3.5.16 <u>Lemma</u>: If $a \in A$ and $e,f \in F^+$ then $(a\psi_f)\psi_e\tau(e,f) = \tau(e,f)a\psi_{e+f}$.

<u>Proof</u>: If $e = f = k$ the result is immediate since $\tau(k,k) = (k,1)$, by Lemma 3.5.11. If $e = k$ and $f \neq k$, since $\tau(k,f) = (k,1)$ by Lemma 3.5.11, we again have the result. Similarly, if $e \neq k$ and $f = k$ the result holds. We are thus left to consider the case where $e \neq k$ and $f \neq k$. From (3.5.14), $u_e(u_f a) = u_e(a\psi_f)u_f = (a\psi_f)\psi_e u_e u_f = (a\psi_f)\psi_e\ \tau(e,f)u_{e+f}$, by (3.5.10) also. On the other hand, by (3.5.10), $(u_e u_f)a = \tau(e,f)u_{e+f}a$ and, by (3.5.14), $(u_e u_f)a = \tau(e,f)(a\psi_{e+f})u_{e+f}$. Thus $(a\psi_f)\psi_e\ \tau(e,f)u_{e+f} = \tau(e,f)(a\psi_{e+f})u_{e+f}$. By postmultiplying by u^{-1}_{e+f} and noting that $u_{e+f}u^{-1}_{e+f} = (k,1)$ we have the result.

We return now to the theorem. The functions $\tau: F^+ \times F^* \to H_{(k,1)}$ and $\psi: F^+ \to \text{End } A$ defined above are shown in Lemmas 3.5.9 3.5.11, 3.5.15 and 3.5.16 to satisfy the necessary requirements for the construction of the semigroup $S(F, k, \Phi; A, \tau, \psi)$. We are now in a position to verify that $\sigma: S \to S(F, k, \Phi; A, \tau, \psi)$ is a homomorphism.

Let $s \in S\setminus\{0\}$. Then $s0 = 0$ and $(s0)\sigma = 0 = (s\sigma)(0\sigma)$. If $s,t \in S\setminus\{0\}$ with $st = 0$ then $(st)\sigma = 0$. Let $s\sigma = (e,g_\alpha,f)$ and $t\sigma = (p,h_\beta,q)$. Then $(s\sigma)(t\sigma) = (e,g_\alpha,f)(p,h_\beta,q) = (e,g_\alpha,f)(f,\alpha,f)(p,\beta,p) \times (p,h_\beta,q)$. However $(f,\alpha)(p,\beta) = s^{-1}stt^{-1} = 0$ and so $(f,\alpha,f)(p,\beta,p)$

$= (H_{(f,\alpha)})\phi(H_{(p,\beta)})\phi = (H_{(f,\alpha)(p,\beta)})\phi = 0$ since $\mathcal{H}$ is a congruence on S and ϕ is a homomorphism. Thus $(s\sigma)(t\sigma) = 0 = (st)\sigma$.

Let $s,t \in S\setminus\{0\}$ with $st \neq 0$. Let $s\sigma = (e,g_\alpha,f)$ and $t\sigma =$ (p,h_β,q). From this we have $s = u_e^{-1}g_\alpha u_f$ and $t = u_p^{-1}h_\beta u_q$. Hence $st = u_e^{-1}g_\alpha u_f u_p^{-1}h_\beta u_q$. Applying Lemma 3.5.12 we have $u_f u_p^{-1} = u_{fp-f}^{-1}\tau(fp-f,f)(\tau(fp-p,p))^{-1}u_{fp-p}$ and so $st = u_e^{-1}g_\alpha u_{fp-f}^{-1}\tau(fp-f,f) \times (\tau(fp-p,p))^{-1}u_{fp-p}h_\beta u_q$. From 3.5.14, $u_{fp-f}g_\alpha^{-1} = (g_\alpha^{-1}\psi_{fp-f})u_{fp-f}$ and $u_{fp-p}h_\beta = (h_\beta\psi_{fp-p})u_{fp-p}$ so that $st = u_e^{-1}((g_\alpha^{-1}\psi_{fp-f})u_{fp-f})^{-1} \times \tau(fp-f,f)(\tau(fp-p,p))^{-1}(h_\beta\psi_{fp-p})u_{fp-p}u_q = u_e^{-1}u_{fp-f}^{-1}(g_\alpha\psi_{fp-f})\tau(fp-f,f) \times (\tau(fp-p,p))^{-1}(h_\beta\psi_{fp-p})u_{fp-p}u_q$, since ψ_{fp-f} is a homomorphism. Applying (3.5.10) we now have $st = (\tau(fp-f,e)u_{(fp-f)+e})^{-1}g_\alpha\psi_{fp-f} \times \tau(fp-f,f)(\tau(fp-p,p))^{-1}(h_\beta\psi_{fp-p})\tau(fp-p,q)u_{(fp-p)+q}$. However $(H_{st})\phi = (H_s)\phi(H_t)\phi = (e,\alpha,f)(p,\beta,q)$ since $\mathcal{H}$ is a congruence on S and ϕ is a homomorphism. Thus $(H_{st})\phi = ((fp-f)+e,\gamma,(fp-p)+q)$ where

$$\gamma = \begin{cases} \alpha\beta & \text{if } f = p \\ \alpha & \text{if } f < p \\ \beta & \text{if } f > p \\ 1 & \text{if } f,\ p \text{ are incomparable} \end{cases}$$

It can be readily be checked that in each of these cases $(\tau(fp-f,e))^{-1} \times (g_\alpha\psi_{fp-f})\tau(fp-f,f)(\tau(fp-p,p))^{-1}(h_\beta\psi_{fp-p})\tau(fp-p,q)$ is in G_γ and so, by the definition of σ, $(st)\sigma =$

$$((fp-f)+e,\ (\tau(fp-f,e))^{-1}(g_\alpha\psi_{fp-f})\tau(fp-f,f)(\tau(fp-p,p))^{-1}(h_\beta\psi_{fp-p}) \times \tau(fp-p,q),\ (fp-p)+q) = (s\sigma)(t\sigma) \text{ in } S(F,\ k,\ \Phi; A,\ \tau,\ \psi).$$

3.6 Isomorphisms between semigroups of the type $S(E,\ k,\ \Phi;\ A,\ \tau,\theta)$

In this section, following some of the notions of [5, Section 4] we consider certain isomorphisms between semigroups of the type considered above.

3.6.1 <u>Theorem</u>: Let $S = S(E, k, \Phi; A, \tau, \theta)$ and $T = S(F, l, \Psi; B, \sigma, \lambda)$ where A and B are centric inverse monoids.

(a) Let $\psi: S \to T$ be an isomorphism, for which $(k,1_A,k)\psi = (\ell,1_B,\ell)$ where 1_A is the identity of A and 1_B is the identity of B. Then there exist isomorphisms $\alpha: A \to B$ and $\beta: E \to F$ with $k\beta = \ell$, and, for all $e \in E^*$, there exists $x_e \in H_{1_B}$, with $x_k = 1_B$, such that $(e,a,f)\psi = (e\beta, x_e^{-1}(a\alpha)x_f, f\beta)$. The following conditions are also satisfied:- (3.6.1)(i) $(e+f)\beta = e\beta + f\beta$ for all $e \in E^+$, $f \in E^*$

(3.6.1)(ii) $(x_e)x_f\lambda_{e\beta}\sigma(e\beta,f\beta) = (\tau(e,f))\alpha\, x_{e+f}$, for all $e \in E^+$, $f \in E^*$

(3.6.1)(iii) $(a\alpha)\lambda_{e\beta} = x_e^{-1}(a\theta_e)\alpha x_e$ for all $a \in A$, $e \in E^+$.

(b) Conversely, if $\alpha: A \to B$ and $\beta: E \to F$, with $k\beta = \ell$, are isomorphisms and if, for all $e \in E^*$, there exists $x_e \in H_{1_B}$, with $x_k = 1_B$, satisfying conditions (3.6.1)(i)-(iii) then the mapping $\psi: S \to T$ defined by $(e,a,f)\psi = (e\beta, x_e^{-1}(a\alpha)x_f, f\beta)$, and $0\psi = 0$, is an isomorphism.

<u>Proof</u>: As A and B are centric inverse monoids (3.4.2) holds and so $((e,1_A,e),(k,1_A,k)) \in \mathcal{D}$ in S for all $e \in E^*$. Thus $((e,1_A,e)\psi, (k,1_A,k)\psi) \in \mathcal{D}$ in T. As F admits a factorisation compatible with the $\mathcal{D}$-structure of T and $(k,1_A,k)\psi = (\ell,1_B,\ell)$ we have $(e,1_A,e)\psi = (n,1_B,n)$ (say). Let $\beta: E \to F$ be defined as follows: $0\beta = 0$ and $(e,1_A,e)\psi = (e\beta,1_B,e\beta)$ for all $e \in E^*$. Since ψ is an isomorphism we can deduce that β is a bijection. Also if $e,f \in E^*$ with $ef = 0$ then $((e,1_A,e)(f,1_A,f))\psi = 0\psi = 0$ and so $(e,1_A,e)\psi(f,1_A,f)\psi = 0$ so that $(e\beta,1_B,e\beta)(f\beta,1_B,f\beta) = 0$ and we have $(e\beta)(f\beta) = 0 = (ef)\beta$. If $e,f \in E^*$ with $ef \neq 0$ then

$((e,1_A,e)(f,1_A,f))\psi = (ef,1_A,ef)\psi = ((ef)\beta,1_B,(ef)\beta)$. Also, as ψ is an isomorphism, $((e,1_A,e)(f,1_A,f))\psi = (e,1_A,e)\psi(f,1_A,f)\psi = (e\beta,1_B,e\beta) \times (f\beta,1_B,f\beta) = ((e\beta)(f\beta),1_B,(e\beta)(f\beta))$ as, since $((e,1_A,e)(f,1_A,f))\psi \neq 0$, $(e\beta)(f\beta) \neq 0$. Hence $(e\beta)(f\beta) = (ef)\beta$ and we have $\beta:E \to F$ an isomorphism with $k\beta = \mathfrak{k}$.

In this next section of the proof we show that if $\delta \varepsilon E_A$ then $(k,\delta,k)\psi = (\mathfrak{k},\gamma,\mathfrak{k})$ for some $\gamma \varepsilon E_B$. Let $(k,\delta,k) \varepsilon S$ with $\delta \varepsilon E_A$. Then by Theorem 3.3.4,(1), (k,δ,k) is an idempotent in S and so $(k,\delta,k)\psi$ is an idempotent in T so that $(k,\delta,k)\psi = (p,\eta,p)$ (say) where $\eta \varepsilon E_B$. We note that, by Theorem 3.3.4 (6), $(k,\delta,k) \leq (k,1_A,k)$ and so $(p,\eta,p) \leq (k,1_A,k)\psi = (\mathfrak{k},1_B,\mathfrak{k})$. Hence, by Theorem 3.3.4,(6), either $p < \mathfrak{k}$ or $p = \mathfrak{k}$ and $\eta \leq 1_B$. Assuming that $p < \mathfrak{k}$, let $(p,1_B,p) = (n,\mu,n)\psi$ (say), where $\mu \varepsilon E_A$. Then as $(p,\eta,p) \leq (p,1_B,p) < (\mathfrak{k},1_B,\mathfrak{k})$ we have $(k,\delta,k) \leq (n,\mu,n) < (k,1_A,k)$. Hence $k \leq n$ and $n \leq k$ so that $k = n$. However $((p,1_B,p),(\mathfrak{k},1_B,\mathfrak{k})) \varepsilon \mathcal{D}$ in T by Theorem 3.3.4(7), and so $((n,\mu,n),(k,1_A,k)) \varepsilon \mathcal{D}$ in S so that, by Theorem 3.3.4(7) again, we have $\mu = 1_A$ and so $(n,\mu,n) = (k,1_A,k)$ which is a contradiction. Thus $p \nless \mathfrak{k}$. Hence $p = \mathfrak{k}$ and $(k,\delta,k)\psi = (\mathfrak{k},\eta,\mathfrak{k})$ is of the required form.

Using the above we obtain the following result. Let $a \varepsilon A$; then $((k,a,k),(k,aa^{-1},k)) \varepsilon \mathcal{H}_S$ and so $((k,a,k)\psi,(\mathfrak{k},\gamma,\mathfrak{k})) \varepsilon \mathcal{H}_T$ (say). Thus $(k,a,k)\psi = (\mathfrak{k},b,\mathfrak{k})$ where $b \varepsilon B$. We define $\alpha:A \to B$ as follows:- $(k,a,k)\psi = (\mathfrak{k},a\alpha,\mathfrak{k})$. Clearly α is a bijection. Also if $a_1,a_2 \varepsilon A$ then $(k,a_1a_2,k)\psi = (\mathfrak{k},(a_1a_2)\alpha,\mathfrak{k})$. However $(k,a_1a_2,k) = (k,a_1,k)(k,a_2,k)$ so that, as ψ is an isomorphism, $(k,a_1a_2,k)\psi = (k,a_1,k)\psi(k,a_2,k)\psi = (\mathfrak{k},a_1\alpha,\mathfrak{k})(\mathfrak{k},a_2\alpha,\mathfrak{k}) = (\mathfrak{k},(a_1\alpha)(a_2\alpha),\mathfrak{k})$ and we have $(a_1a_2)\alpha = (a_1\alpha)(a_2\alpha)$, so that α is an isomorphism.

Let $e \varepsilon E^*$. Then $((k,1_A,e),(k,1_A,k)) \varepsilon \mathcal{R}_S$ and $((k,1_A,e),(e,1_A,e)) \varepsilon \mathcal{L}_S$. Thus $((k,1_A,e)\psi,(\mathfrak{k},1_B,\mathfrak{k})) \varepsilon \mathcal{R}_T$ and $((k,1_A,e)\psi,(e\beta,1_B,e\beta)) \varepsilon \mathcal{L}_T$ so that $(k,1_A,e)\psi = (\mathfrak{k},x,e\beta)$ for some $x \varepsilon A$. Denote by x_e the element in $\mathcal{B}$ such that $(k,1_A,e)\psi = (\mathfrak{k},x_e,e\beta)$. From the Green's relations above we also have $x_e x_e^{-1} = 1_B$ so that $x_e \varepsilon\ H_{1_B}$. Also $x_k = 1_B$ as $(k,1_A,k)\psi = (\ell,1_B,\ell)$.

Let $(e,a,f) \varepsilon S$; then $(e,a,f) = (e,1,k)(k,a,k)(k,1,f) = (k,1,e)^{-1} \times (k,a,k)(k,1,f)$. Thus $(e,a,f)\psi = ((k,1,e)\psi)^{-1}(k,a,k)\psi(k,1,f)\psi$ as ψ is an isomorphism. Hence $(e,a,f)\psi = (\ell,x_e,e\beta)^{-1}(\ell,a\alpha,\ell)(\ell,x_f,f\beta) = (e\beta,x_e^{-1},\ell)(\ell,a\alpha,\ell)(\ell,x_f,f\beta) = (e\beta,x_e^{-1}(a\alpha)x_f,f\beta)$.

To complete the proof of (a) we need only check that conditions 3.6.1 (i), (ii) and (iii) are satisfied. We note that if $e \varepsilon E^+$ and $f \varepsilon E^*$ then $(e,1_A,e)(k,1_A,f) = (e,(\tau(e,k))^{-1}\tau(e,f),e+f) = (e,\tau(e,f), e+f)$. Hence $((e,1_A,e)(k,1_A,f))\psi = (e\beta,x_e^{-1}(\tau(e,f))\alpha x_{e+f},(e+f)\beta)$. However, ψ is an isomorphism and $((e,1_A,e)(k,1_A,f))\psi = (e,1_A,e)\psi\ (k,1_A,f)\psi = (e\beta,1_B,e\beta)(\ell,x_f,f\beta) = (e\beta,(\sigma(e\beta,1))^{-1}\ x_f\ \lambda_{e\beta}\sigma(e\beta,f\beta),e\beta + f\beta)$. Thus $(e+f)\beta = e\beta + f\beta$, equating the third terms in $((e,1_A,e)(k,1_A,f)\psi$ and $(e,1_A,e)\psi(k,1_A,f)\psi$, so that condition 3.6.1 (i) is satisfied. Equating the middle terms we have $x_e^{-1}(\tau(e,f))\alpha x_{e+f} = (\sigma(e\beta,1))^{-1}(x_f)\lambda_{e\beta}\sigma(e\beta,f\beta) = (x_f)\lambda_{e\beta}\sigma(e\beta,f\beta)$. Pre-multiplying both these expressions by x_e and noting that $x_e x_e^{-1} = 1_B$ we have $(\tau(e,f))\alpha\ x_{e+f} = x_e(x_f)\lambda_{e\beta}\sigma(e\beta,f\beta)$ which is condition 3.6.1 (ii). To obtain condition 3.6.1 (iii) we consider $(k,1_A,e)(k,a,k)$ where $a \varepsilon A$ and $e \varepsilon E^+$. We have $(k,1_A,e)(k,a,k) = (k,a\theta_e,e)$ so that $((k,1_A,e)(k,a,k))\psi = (k,a\theta_e,e)\psi = (\ell,(a\theta_e)\alpha x_e,e\beta)$. However, as ψ is an isomorphism we have $((k,1_A,e)(k,a,k))\psi = (k,1_A,e)\psi \times (k,a,k)\psi = (\ell,x_e,e\beta)(\ell,a\alpha,\ell) = (\ell,x_e(a\alpha)\lambda_{e\beta},e\beta)$. Equating the middle terms of $((k,1_A,e)(k,a,k))\psi$ and $(k,1_A,e)\psi(k,a,k)\psi$ we have $(a\theta_e)\alpha x_e = x_e(a\alpha)\lambda_{e\beta}$. On pre-multiplying both these expressions by x_e^{-1} and noting that $x_e^{-1}x_e = 1_B$ we have $(a\alpha)\lambda_{e\beta} = x_e^{-1}(a\theta_e)\alpha x_e$ which is condition 3.6.1 (iii).

(b) We first show that if ψ is as defined then ψ is a bijection. Let (e,a,f), $(g,b,h) \varepsilon S$ with $(e,a,f)\psi = (g,b,h)\psi$. Then $(e\beta,x_e^{-1}(a\alpha)x_f, f\beta) = (g\beta,x_g^{-1}(b\alpha)x_h,h\beta)$. Thus $e\beta = g\beta$ and $f\beta = h\beta$ so that, since β is an isomorphism $e = g$ and $f = h$. Also $x_e^{-1}(a\alpha)x_f = x_g^{-1}(b\alpha)x_h = x_e^{-1}(b\alpha)x_f$. Hence as $x_e x_e^{-1} = 1_B = x_f x_f^{-1}$ we have $a\alpha = b\alpha$ so that, as α is an isomorphism, $a = b$. Thus ψ is injective. Also let $(m,c,n) \in T$. Then as $\beta: E \to F$ is an isomorphism, there exists $e,f \in E$ such that $e\beta = m$ and $f\beta = n$. We note that $x_e c x_f^{-1} \in B$ and so, since $\alpha: A \to B$ is an isomorphism, there exists $a \in A$ such that $a\alpha = x_e c x_f^{-1}$ i.e. $x_e^{-1}(a\alpha)x_f = c$. Thus $(e,a,f)\psi = (m,c,n)$ and we have that ψ is surjective.

We must now verify that ψ is a homomorphism. Let $(e,a,f),(g,b,h) \in S$ with $fg = 0$. Then $(e,a,f)(g,b,h) = 0$ and $((e,a,f)(g,b,h))\psi = 0$. Also $(fg)\beta = (f\beta)(g\beta) = 0$ as β is an isomorphism and so $(e,a,f)\psi(g,b,h)\psi = (e\beta, x_e^{-1}(a\alpha)x_f, f\beta)(g\beta, x_g^{-1}(b\alpha)x_h, h\beta) = 0$ and we have $((e,a,f)(g,b,h))\psi = 0 = (e,a,f)\psi(g,b,h)\psi$. If $(e,a,f),(g,b,h) \in S$ with $fg \neq 0$ then $(e,a,f) \times (g,b,h) = ((fg-f)+e, uv, (fg-g)+h)$ where $u = (\tau(fg-f,e))^{-1} a\theta_{fg-f}\tau(fg-f,f)$ and $v = (\tau(fg-g,g))^{-1} b\theta_{fg-g}\tau(fg-g,h)$. Thus $((e,a,f)(g,b,h))\psi = ((fg-f)\beta + e\beta, x^{-1}_{(fg-f)+e}(uv)\alpha\, x_{(fg-g)+h}, (fg-g)\beta + h\beta)$. We note that $(uv)\alpha = (u\alpha)(v\alpha)$ as α is an isomorphism and also $u\alpha = ((\tau(fg-f,e))\alpha)^{-1} \times (a\theta_{fg-f})\alpha(\tau(fg-f,f))\alpha$. By applying conditions 3.6.1(ii) and (iii) we have $u\alpha = (x_{fg-f}(x_e\lambda_{(fg-f)\beta})\sigma((fg-f)\beta,e\beta)x^{-1}_{(fg-f)+e})^{-1} \times x_{fg-f}(a\alpha)\lambda_{(fg-f)\beta}x^{-1}_{fg-f}\, x_{fg-f}(x_f\lambda_{(fg-f)\beta})\sigma((fg-f)\beta,f\beta)\, x^{-1}_{fg-f+f}$ $= x_{(fg-f)+e}(\sigma((fg-f)\beta,e\beta))^{-1}\, x_e^{-1}\lambda_{(fg-f)\beta}(a\alpha)\lambda_{(fg-f)\beta}(x_f)\lambda_{(fg-f)\beta} \times \sigma((fg-f)\beta,f\beta)\, x^{-1}_{fg} = x_{(fg-f)+e}(\sigma((fg-f)\beta,e\beta))^{-1}(x_e^{-1}(a\alpha)x_f)\lambda_{(fg-f)\beta} \times \sigma((fg-f)\beta,f\beta)\, x^{-1}_{fg}$, as $\lambda_{(fg-f)\beta}$ is a homomorphism of B. Also, similarly, we obtain $v\alpha = x_{fg}(\sigma(fg-g)\beta,g\beta))^{-1}(x_g^{-1}(b\alpha)x_h)\lambda_{(fg-g)\beta} \times \sigma((fg-g)\beta,h\beta)\, x^{-1}_{(fg-g)+h}$. Hence $((e,a,f)(g,b,h))\psi = ((fg-f)\beta + e\beta,$ $(\sigma((fg-f)\beta,e\beta))^{-1}(x_e^{-1}(a\alpha)x_f)\lambda_{(fg-f)\beta}\sigma((fg-f)\beta,f\beta) \times$ $(\sigma((fg-g)\beta,g\beta))^{-1}(x_g^{-1}(b\alpha)x_h)\lambda_{(fg-g)\beta}\sigma((fg-g)\beta,h\beta), (fg-g)\beta + h\beta)$.

On the other hand, $(e,a,f)\psi(g,b,h)\psi = (e\beta, x_e^{-1}(a\alpha)x_f, f\beta)(g\beta, x_g^{-1}(b\alpha)x_h, h\beta)$ $= ((f\beta)(g\beta) - f\beta) + e\beta, pq, ((f\beta)(g\beta) - g\beta) + h\beta)$ where

$p = (\sigma((f\beta)(g\beta) - f\beta, e\beta))^{-1}(x_e^{-1}(a\alpha)x_f)\lambda_{(f\beta)(g\beta)-f\beta}\sigma((f\beta)(g\beta)-f\beta, f\beta)$ and $q = (\sigma((f\beta)(g\beta)-g\beta, g\beta))^{-1}(x_g^{-1}(b\alpha)x_h)\lambda_{(f\beta)(g\beta)-g\beta}\sigma((f\beta)(g\beta)-g\beta, h\beta)$. To show that $(e,a,f)\psi(g,b,h)\psi = ((e,a,f)(g,b,h))\psi$ we need only show that $(f\beta)(g\beta)-f\beta = (fg-f)\beta$ and $(f\beta)(g\beta)-g\beta = (fg-g)\beta$. As β is a homomorphism we have $(fg)\beta = (f\beta)(g\beta)$. Also, by condition 3.6.1(i) $(fg)\beta = ((fg-g)+g)\beta = (fg-g)\beta + g\beta$ so that $(fg-g)\beta = (fg)\beta - g\beta$ and similarly, $(fg)\beta = ((fg-f)+f)\beta = (fg-f)\beta + f\beta$ so that $(fg-f)\beta = (fg)\beta - f\beta$ and the result is proved.

We must lastly check that ψ is an isomorphism for which $(k,1_A,k)\psi = (k,1_B,k)$. By the definition of ψ, $(k,1_A,k)\psi = (k\beta, x_k^{-1}(1_A)\alpha x_k, k\beta) = (k, x_k^{-1} 1_B x_k, k) = (k,1_B,k)$.

3.7 Special Cases and Applications

The first special case we consider is that when E is an ω-tree with zero as this is the most complicated case which can actually be computed.

3.7.1 Theorem: Let E be an ω-tree with zero and let A be a centric inverse monoid with identity 1. Fix $k \in E^*$ and let $e \to v_e$ be a mapping of E into H_1 with the property that $v_e = 1$ for all $e \leq k$. Let α be an endomorphism of A into H_1 and let α^0 denote the identity mapping on A. For each pair $(i,x) \in N \times E^*$ define

$$w_{i,x} = \begin{cases} (v_x\alpha^{i-1})(v_{x+1}\alpha^{i-2}) \;\ldots\; (v_{x+i-1}), & \text{if } i \geq 1 \\ 1, & \text{if } i = 0. \end{cases}$$

Let $S = E^* \times A \times E^* \cup \{0\}$ and define multiplication on S as follows:- $(m,a,n)(r,b,s) = (m+t, w_{t,m}^{-1}(a\alpha^t)w_{t,n} w_{u,r}^{-1}(b\alpha^u)w_{u,s}, s+u)$ where $t = [n,nr]$ and $u = [r,nr]$, if $nr \neq 0$, and all other products are zero. Then S is a 0-simple inverse semigroup whose semilattice admits a factorisation compatible with the $\mathcal{D}$-structure of S.

Proof : The proof consists of showing that S, as described above, is in fact of the form $S(E, k, \Phi; A, \tau, \theta)$ and the result then follows immediately from Theorems 3.3.3 and 3.3.4, and (3.4.2).

The first feature we concentrate on is an addition on E. Since E is an ω-tree with zero each principal ideal of E is an ω-chain with zero and so is inversely well-ordered. Hence, by the note following [6, Theorem 3.2], $\mathcal{H} = i$ on T_E. Thus, if $e,f \in E^*$, there exists a unique isomorphism $\xi_{e,f}:Ee \to Ef$ as described in (1.3.10). If $k \in E^*$ is fixed, an addition Φ with identity k can be defined on E whereby $e+f = e\xi_{k,f}$ for all $e \in E^+$, $f \in E^*$. In the product $(m,a,n) \times (r,b,s)$ we need to reconcile $m+t$, where $t = [n,nr]$, and $s+u$, where $u = [r,nr]$ with $(nr-n)+m$ and $(nr-r)+s$, respectively. We note that $n+t = nr$ so that $m+t = (nr)\xi_{n,k}\xi_{k,m} = (nr\xi_{k,n}^{-1})\xi_{k,m} = (nr-n)+m$. Similarly $s+u = (nr-r)+s$.

We next note that $E^+ = \{k+i : i \in N\}$. Let $w_{i,f} = \tau(k+i,f)$ for all $i \in N$, $f \in E^*$. Then $\tau : E^+ \times E^* \to H_1$ and immediately satisfies condition 3.2.1,(1). We also make the definition that $\theta_{k+i} = \alpha^i$ for all $i \in N$ and set to checking that conditions 3.2.1, (2) and (3) are satisfied. For condition 3.2.1, (2), $\tau(k+i,k+j)\tau(k+i+j,g) = \tau(k+i+j,g)$ as $\tau(k+i,k+j) = 1$, since $k+j \leq k$, and so $v_{k+j+n} = 1$ for all $n \in N$. On the other hand, if $i \geq 1$ and $j \geq 1$, $(\tau(k+j,g))\theta_{k+i}\tau(k+i,g+j) = w_{j,g}\alpha^i w_{i,g+j} = ((v_g\alpha^{j-1})\ldots v_{g+j-1})\alpha^i \times (v_{g+j}\alpha^{i-1})\ldots v_{g+j+i-1} = w_{i+j,g} = \tau(k+i+j,g)$. This still holds if $i=0$ or $j=0$. Thus condition 3.2.1, (2) is satisfied. For condition 3.2.1 (3) consider $\tau(k+i,k+j) \times (a\theta_{k+i+j}) = a\alpha^{i+j}$ since, as above, $\tau(k+i,k+j) = 1$. Also $(a\theta_{k+j})_{k+i} \times \tau(k+i,k+j) = (a\alpha^j)\alpha^i 1 = a\alpha^{i+j}$ and we have condition 3.2.1,(3) satisfied.

We note now that, as E is an ω-tree with zero, E contains no primitive idempotents and so, by Theorem 3.3.3 (a), S is 0-simple. The remainder of the result follows directly from Theorem 3.3.4 and (3.4.2).

3.7.2 There is also a converse to this result.

3.7.3 <u>Theorem</u>: Let S be a 0-simple inverse semigroup whose semilattice E admits a factorisation compatible with the $\mathcal{D}$-structure of S. Let $E^* = F^* \times Y$, as in 3.4.1, where F is an ω-tree with zero. Then S has the form described in Theorem 3.7.1.

Proof: The conditions of Theorem 3.5.7 are satisfied by S. Thus there is an associative addition defined on F with $e+f = e\xi_{k,f}$ where k is an arbitrary fixed element of F*. We have shown above, in Theorem 3.7.1, that if $n,r,m \in F^*$ with $nr \neq 0$, then, if $t = [n,nr]$, $m+t = (nr-n)+m$. We define A as in Theorem 3.5.7.

We select a set of representatives u_f of the $\mathcal{H}$-classes of S contained in $R_{(k,1)}$ as in the proof of Theorem 3.5.7, with the following provisos:
Let u be the representative of H_a, where $(H_a)\phi = (k,1,k+1)$ and let u^n be the representative of H_x, where $(H_x)\phi = (k,1,k+n)$ for all $n \in N$ with $n \geq 1$, taking ϕ as in Theorem 3.5.7. Define τ, ψ as in Theorem 3.5.7. Thus we have $u_{k+n}u_f = \tau(k+n,f)u_{f+n}$, for all $f \in F^*$ and $n \in N$, so that $u^n u_f = \tau(k+n,f)u_{f+n}$. Define $\tau(k+n,f) = w_{n,f}$ for all $n \in N$, $n \geq 1$, and for all $f \in F^*$, and let $v_f = w_{1,f}$ for all $f \in F^*$. Write $w_{0,f} = 1$ for all $f \in F^*$. We also have $ua = (a\psi_{k+1})u$, for all $a \in A$. Let $\psi_{k+1} = \alpha$, so that $\psi_{k+n} = \alpha^n$ for all $n \in N$, $n \geq 1$. Let $\alpha^0 = \psi_k$, the identity automorphism on A.

From Theorem 3.5.7 we have that S is of the form $S(F,k,\phi{:}A,\tau,\psi)$. From the results obtained above we have $S \cong F^* \times A \times F^* \cup \{0\}$ with the following multiplication in $F^* \times A \times F^* \cup \{0\}$:-

$(m,a,n)(p,b,q) = (m+t, w_{t,m}^{-1}(a\alpha^t)w_{t,n}\, w_{u,p}^{-1}(b\alpha^u)w_{u,q}, q+u)$ where $t = [n,np]$ and $u = [p,np]$, if $np \neq 0$; all other products are zero.

The mapping α satisfies the requirements of Theorem 3.7.1 and we now need to check that v_e and $w_{i,e}$ are as required. We note that if $e \in F^+$ then $v_e = v_{k+i}$ for some $i \in N$ and v_e is such that $uu_{k+i} = v_e u_{k+i+1}$, i.e. $u^{i+1} = v_e u^{i+1}$. Thus we have $v_e = (k,1)$. Examining $w_{i,f}$ we see that $w_{1,f} = v_f$ for all $f \in F^*$. We assume that for $i = p$, $w_{p,f} = v_f\alpha^{p-1}\, v_{f+1}\alpha^{p-2} \ldots v_{f+p-1}$ is true. We have $u^{p+1}u_f = (w_{p+1,f})\, u_{f+p+1}$. However $u^{p+1}u_f = u(u^p u_f) = u(w_{p,f}u_{f+p})$ $= (w_{p,f})\alpha u u_{f+p} = (w_{p,f})\alpha w_{1,f+p}\, u_{f+p+1}$. Hence $w_{p+1,f} = (w_{p,f})\alpha w_{1,f+p}$ and the condition regarding $w_{i,f}$ is proved.

3.7.4 Theorem: Let E be a 0-direct union of ω-chains and let A be a centric inverse monoid with identity 1. Let $\alpha : A \to H_1$ be an endomorphism with α^0 the identity mapping on A.

Let $S = [(N\times N)\times(I\times I)\times A]\cup\{0\}$ and define a multiplication on S as follows:- $((m,n),(i,j),a)((p,q),(j,k),b) = ((m-n+t,q-p+t),(i,k), a\alpha^{t-n}\, b\alpha^{t-p})$ where $t = \max\{n,p)$; all other products are zero. Then S is a 0-simple inverse semigroup whose semilattice admits a factorisation compatible with the $\mathcal{D}$-structure of S.

Proof: The proof consists of showing that S has the form of the semigroup described in Theorem 3.7.1 and the result is then immediate. First a 0-direct union of ω-chains is, as described in (2.6.4), of the form $N \times I \cup\{0\}$ and is a special type of ω-tree with zero. The only non zero products in $N \times I$ are those of the form $(n,i)(m,j)$ with $i = j$. Take each $v_e = 1$ in Theorem 3.7.1. With $E^* = N \times I$ the multiplication on S in Theorem 3.7.1 is thus $((m,i),a,(n,j))((p,j), b,(q,h)) = ((m+t,i),a\alpha^t b\alpha^u,(q+u,h))$ where $t = [(n,j),(n,j)(p,j)]$ and $u = [(p,j),(n,j)(p,j)]$. Thus, if $x = \max(n,p)$, $t = x-n$ and $u = x-p$, so that $((m,i),a,(n,j))((p,j),b,(q,h)) = ((m+x-n,i), a\alpha^{x-n}\, b\alpha^{x-p}, (q+x-p,h))$ which is the same product as defined in the statement of Theorem 3.7.4 and so the result is proved.

3.7.5 The converse of this result is as follows:-

Theorem: Let S be a 0-simple inverse semigroup whose semilattice E admits a factorisation compatible with the $\mathcal{D}$-structure of S. Let $E^* = F^* \times Y$, as in 3.4.1, where F is a 0-direct union of ω-chains. Then S has the form described in Theorem 3.7.4.

Proof: Clearly F is an ω-tree with zero and so the conditions of Theorem 3.7.3 are satisfied. We have shown, in the proof of Theorem 3.7.4, that if $(m,i),(n,j),(r,j) \in F^*$ then, if $t = [(m,j),(r,j)(n,j)]$ and $x = \max(n,r)$, we have $(m+t,i) = (m+x-n,i)$.

Fix $(0,z) \in F^*$ and, as in Theorem 3.5.7, choose a set of representatives of the $\mathcal{H}$-classes of S contained in $R_{((0,z),1)}$ as follows:-

Let u be the representative of H_a, where $(H_a)\phi = ((0,z),1,(1,z))$ and let u^n be the representative of H_x, where $(H_x)\phi = ((0,z),1,(n,z))$ for all $n \in N$, $n \geq 1$, and

let i_x be the representative of H_y, where $(H_y)\phi = ((0,z),1,(0,x))$ for all $x \in Y$, and

let $u^n i_x$ be the representative of H_w, where $(H_w)\phi = ((0,z),1,(n,x))$, for all $n \in N$, $n \geq 1$ and all $x \in Y$.

We also stipulate that $i_z = ((0,z),1)$.

With these representatives we consider the mapping τ defined in Theorems 3.5.7 and 3.7.3. Firstly we note that $F^+ = \{(n,z): n \in N\}$ and so for all $(n,z) \in F^+$ and $(m,x) \in F^*$ we have $u_{(n,z)}u_{(m,x)} = \tau((n,z),(m,x))u_{(n+m,x)}$. However $u_{(n,z)} = u^n$, $u_{(m,x)} = u^m i_x$ and $u_{(n+m,x)} = u^{n+m} i_x$. Hence $u^n u^m i_x = \tau((n,z),(m,x))u^{n+m} i_x$, from which we immediately have that $\tau((n,z),(m,x)) = ((0,z),1)$. Hence, for all $(n,x) \in F^*$ we have $v_{(n,x)} = \tau((1,z),(n,x)) = ((0,z),1)$. Using this result in Theorem 3.7.3 we thus have a semigroup as described in Theorem 3.7.4.

3.7.6 <u>Theorem</u>: Let A be a centric inverse monoid with identity element 1. Let $\theta: A \to H_1$ be an endomorphism with θ^0 denoting the identity automorphism on A.

Let $S = (N \times N \times A) \cup \{0\}$ and define multiplication on S as follows:- $(m,n,a)(p,q,b) = (m-n+t, q-p+t, a\theta^{t-n} b\theta^{t-p})$ where $t = \max(n,p)$; all other products are zero. Then S is a 0-simple inverse semigroup whose semilattice admits a factorisation compatible with the $\mathcal{D}$-structure of S.

Proof: This is a modification of Theorem 3.7.4 since an ω-chain with zero is isomorphic to $N \cup \{0\}$ and an ω-chain with zero is, trivially, a 0-direct union of ω-chains. In fact we have exactly the situation of Theorem 3.7.4 with $|I| = 1$. The theorem follows immediately.

3.7.7 Theorem: Let S be a 0-simple inverse semigroup whose semilattice E admits a factorisation compatible with the $\mathcal{D}$-structure of S. Let $E^* = F^* \times Y$ where F is an ω-chain with zero. Then S has the form described in Theorem 3.7.6.

Proof: Since an ω-chain with zero is a 0-direct union of ω-chains S satisfies Theorem 3.7.5. However $F^* = N$ in this case, i.e. $|I| = 1$ and so we have exactly the form described in Theorem 3.7.6.

The results of Theorems 3.7.6 and 3.7.7 were obtained by Munn in [11, Theorem 3.3].

The special cases obtained so far have been obtained by successive modifications of E in $S(E, k, \mathcal{P}; A, \tau, \theta)$. We obtain a further special case by taking E to be an ω-chain with zero as in Theorem 3.7.6 and, in addition, taking E_A to be a finite chain.

3.7.8 Theorem: Let A be a centric inverse monoid with identity element 1 whose semilattice is a finite chain. Let $\theta: A \to H_1$ be an endomorphism with θ^0 denoting the identity mapping on A.

Let $S = (N \times N \times A) \cup \{0\}$ and define multiplication on S as follows:- $(m,n,a)(p,q,b) = (m-n+t, q-p+t, a\theta^{t-n}\, b\theta^{t-p})$ where $t = \max(n,p)$; all other products are zero. Then S is a 0-simple inverse ω-semigroup.

Conversely, every 0-simple inverse ω-semigroup is of this form.

Proof: Clearly S satisfies the conditions of Theorem 3.7.6 and so is a 0-simple inverse semigroup. By Theorems 3.7.6 and 3.3.4, and (3.4.2) E_S admits a factorisation and $E_S^* = N \times E_A$. However, if $E_A = \{e_1 = 1 > e_2 > \ldots > e_d\}$ (say) then $N \times E_A = \{(0,e_1) > (0,e_2) \ldots > (0,e_d) > (1,e_1) > (1,e_2) > \ldots (1,e_d) > \ldots\}$. Thus $N \times E_A$ is itself an ω -chain and so, in the terminology of [7], S is an ω-semigroup with zero.

Conversely, let S be a 0-simple inverse ω-semigroup with semilattice E where $E^* = \{e_i : i \in N \text{ and } e_i > e_j \iff i < j\}$. By [7, Lemma 4.3 (iii)] there exists $d \in N$, $d \geq 1$, the number of non zero $\mathcal{D}$-classes of S, such that $(e_i, e_j) \in \mathcal{D} \iff i \equiv j \pmod{d}$. We consider the mapping $e_i \to (n,s)$ where $i = nd+s$ and $0 \leq s < d$, $n \in N$. Thus, if $e_i \to (n,s)$ and $e_j \to (m,t)$ then $(e_i, e_j) \in \mathcal{D} \iff s = t$. Also $e_i < e_j \iff i > j$, i.e. $nd+s > md+t$. However $nd+s > md+t \iff n > m$ or $n = m$ and $s > t$. Hence we see that $N \times \{0,1,2,\ldots, d-1\}$ is a factorisation of E_S compatible with the $\mathcal{D}$-structure of S. From this we have that S satisfies the conditions of Theorem 3.7.7 and the result follows.

This result was obtained in [2] by Kochin and in [7] by Munn.

The final simplification in this pattern is to take $|E_A| = 1$, so that A is a group. The following is then the case:-

3.7.9 Theorem: Let A be a group and let α be an endomorphism of A with α^0 denoting the identity automorphism on A.

Let $S = (N \times N \times A) \cup \{0\}$ and define multiplication on S as follows:-

$(m,n,a)(p,q,b) = (m-n+t, q-p+t, a\alpha^{t-n} b\alpha^{t-p})$ where $t = \max(n,p)$: all other products are zero. Then S is a 0-bisimple inverse ω-semigroup.

Conversely, every 0-bisimple inverse ω-semigroup is of this form.

This result was established by Reilly in [12, Theorem 3.5].

Another chain of special cases can be obtained by returning to the original $S(E, k, \mathcal{J}; A, \tau, \theta)$ and taking A to be a group.

3.7.10 <u>Theorem:</u> In $S(E, k, \mathcal{J}; A, \tau, \theta)$ let A be a group. Then $S(E, k, \mathcal{J}; A, \tau, \theta)$ is a 0-bisimple inverse semigroup.

This is immediate from Theorems 3.3.4 (3) and (4). It is the special case of McAlister's result stated in Corollary 3.1.3.

If we now return to Theorems 3.7.1 and 3.7.3 and make the additional modification that $|E_A| = 1$ we have the following result.

3.7.11 <u>Theorem:</u> Let E be an ω-tree with zero and let G be a group with identity 1. Fix $k \in E^*$ and let $e \to v_e$ be a mapping of $E^* \to G$ with the property that $v_e = 1$ for all $e \le k$. Let α be an endomorphism of G with α^0 denoting the identity automorphism on A. For each pair $(i,x) \in N \times E^*$ define $w_{i,x} = \begin{cases} v_x\alpha^{i-1} \cdots v_{x+i-1} & (i \ge 1) \\ 1 & (i = 0). \end{cases}$

Let $S = (E^* \times G \times E^*) \cup \{0\}$ and define a multiplication on S as follows:- $(m,a,n)(r,b,s) = (m+t,\ w_{t,m}^{-1}\, a\alpha^t\, w_{t,n}\, w_{u,r}^{-1}\, b\alpha^u\, w_{u,s}, s+u)$ where $t = [n,nr]$ and $u = [r,nr]$ if $nr \ne 0$; all other products are zero. Then S is a 0-bisimple inverse semigroup whose semilattice is an ω-tree with zero.

Conversely, if S is a 0-bisimple inverse semigroup whose semilattice is an ω-tree with zero, then S has the form described above.

<u>Proof:</u> Clearly S has the form described in Theorem 3.7.1 and so is of the form $S(E, k, \mathcal{J}; A, \tau, \theta)$ where E is an ω-tree with zero and A is a group. Thus by Theorem 3.7.1, S is a 0-simple inverse semigroup. Also, by Theorem 3.3.4 and (3.4.2) the semilattice of S is an ω-tree with zero and, applying Theorem 3.3.4, (4) is 0-bisimple.

Conversely, if S is a 0-bisimple inverse semigroup whose semilattice is an ω-tree with zero, E_S can be considered as having a factorisation $E_S^* \times \{1\}$ which is compatible with the $\mathcal{P}$-structure of S. Thus by Theorem 3.7.3 the result follows.

This result was stated by McAlister in [5, Theorem 6.1].

3.7.12 This modification brings us to a result stated in Theorem 2.6.15 and spotlights the overlap of the situations described in Chapter 2 and Chapter 3. The results deduced from Theorem 2.6.15 follow automatically from Theorem 3.7.11.

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APPENDIX

Revised version of the proof of Theorem 2.6.2(ii) and of 2.6.3.

2.6.2 Theorem:

Proof: (ii) From Theorem 2.5.3, S is of the form $S(E,T,k,G_i,\gamma_i,e,v_f)$. Since S splits over $\mathcal{H}$ there exists a set of representatives A of the $\mathcal{H}$-classes of S which forms a subsemigroup of S. Assume that $A = \{r(f,g) : f,g \in E^*\}$, where, in the notation of(2.5.2), $(H_{r(f,g)})\phi = (f,g)$. Then $r(f,g)r(p,q) = r((f,g)(p,q))$.....2.6.2(a)

This follows as the set of representatives is a subsemigroup of S and $(H_{r(f,g)r(p,q)})\phi = (f,g)(p,q)$. Also $r(f,g)(r(f,g))^{-1} = f$ and $(r(f,g))^{-1}r(f,g) = g$ ....2.6.2(b).

It is also immediate that $(r(f,g))^{-1} = r(g,f)$ ....2.6.2(c).

This follows as $r(f,g)r(g,f)r(f,g)= r((f,g)(g,f)(f,g))$, by 2.6.2(a), i.e. $r(f,g)r(g,f)r(f,g) = r(f,g)$.

For all $f \in E^*$ let $u_f = r(e+i,f)$ where $\underline{f} = i$. We now show that these elements u_f satisfy conditions (a),(b),(c) and (d) of Theorem 2.5.3.

(a) Let $i \in N$ with $0 \leq i \leq k-1$ and let $f \in E^*$ with $\underline{f} = i$. Then $u_{e+i}u_f = r(e+i,e+i)r(e+i,f) = r(e+i,f)$ by 2.6.2(a) and so $u_{e+i}u_f = u_f$ and $u_{e+i} = e+i$.

(b) Let $n \in N$ with $n \geq 1$. Then $u^n_{e+k} = (r(e,e+k))^n = r((e,e+k)^n) = r(e,e+nk) = u_{e+nk}$.

(c) Let $m,n \in N$ with $n \geq 1$ and $0 \leq m \leq k-1$. Then $u_{e+m+nk} = r(e+m,e+m+nk) = r((e+m,e+m)(e,e+nk)) = r(e+m,e+m)r(e,e+nk) = (e+m)u^n_{e+k}$.

(d) Let $f,g \in E^*$ with $\underline{f} = \underline{g} = i$. Then $u_f^{-1}u_g = (r(e+i,f))^{-1}r(e+i,g) = r(f,e+i)r(e+i,g)$ by 2.6.2(c). Thus $u_f^{-1}u_g = r((f,e+i)(e+i,g)) = r(f,g)$.

With the notation of Theorem 2.5.3 we now examine the definition of $m_{t,f}$. We have $m_{t,f}u_{f+t} = u_{e+i+t}u_f$ where $\underline{f} = i$. Let $p = \underline{f+t}$ then we have $m_{t,f}r(e+p,f+t) = r(e+p,e+i+t)r(e+i,f) = r((e+p,e+i+t)(e+i,f))$ by 2.6.2(a). Thus $m_{t,f}r(e+p,f+t) = r(e+p,f+t)$. Hence $m_{t,f} = e+p$, the identity of the group $G_{\underline{f+t}}$. From this we see that for all $f \in E^*$, $v_f = m_{1,f}$ is the identity of the group $G_{\underline{f}+1}$.

2.6.3 From the above theorem we have a necessary and sufficient condition for a 0-simple inverse semigroup whose semilattice is an ω-tree with zero to split over $\mathcal{H}$: namely that it be isomorphic to a semigroup of the form $S(E,T,k,G_i,\gamma_i,e,v_f)$ where, for all $f \in E^*$, v_f is the identity of the group $G_{\underline{f}+1}$. However a sufficient condition for this to occur is that there exists a set of representatives u_f of certain $\mathcal{H}$-classes of S satisfying conditions (a),(b),(c) and (d) of the proof of Theorem 2.5.3 and such that, for all $f \in E^*$ and all $t \in N$, $u_{e+i+t}u_f = u_{f+t}$, where $\underline{f} = i$.