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Is Neo-Logicism founded on a mistake about Higher-Order Logic? Taking Stock: Hale, Heck, and Wright

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Abstract

Four philosophical concerns about higher-order logic in general and the specific demands placed on it by the neo-logicist project are distinguished. The paper critically reviews recent responses to these concerns by, respectively, the late Bob Hale, Richard Kimberly Heck and myself. It is argued that these score some successes. The main aim of the paper, however, is to argue that the most serious objection to the applications of higher-order logic required by the neo-logicist project has not been properly understood. The paper concludes by outlining a strategy, prefigured in recent work of Øystein Linnebo, for meeting this objection.

Foreword

The central ideas of this paper were presented in a keynote address at *Necessary Beings: A Conference in Memory of Bob Hale* which was held at the London Institute of Philosophy on October 17 and 18, 2018, and at a workshop on *Second-order logic and the question of (Im)-predicativity*, held at the University of Oslo on August 20 and 21, 2019. I am indebted to the participants at the conference and workshop, and to a referee for this journal, for a number of pointed questions the attempt to respond to which has, I believe, resulted in significant improvements. Bob himself was impatient of the idea that any serious philosophical difficulty lurked for neo-logicism in its recourse to the kind of higher order logic its execution demands. This was one of very few key points in the epistemology of neo-logicism about which we were not in agreement. I would love to know how he might have reacted to the discussion to follow.

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Introduction

Frege's logicism—his attempt to reconstruct simple arithmetic and, beyond that, real and complex analysis on the basis of what he conceived as a system of pure logic and a repertoire of suitable definitions — required both the resource of what we now think of as classical second order logic with unrestricted predicate comprehension, and the assumption that so rich and powerful a logic is properly in keeping with logicism's epistemological purposes— that it is "epistemically innocent", as Stewart Shapiro and Alan Weir have put it.[1] These presuppositions are, for the most part, inherited by the contemporary proponents of so-called "neo-Fregeanism", "neo-logicism" or "abstractionism".[2] As is well known, however, Quine (1970) influentially argued that higher order logics are in effect a form of

[1] Shapiro and Weir (2000). Shapiro and Weir did not attempt to characterise what should count exactly as "epistemic innocence", contending, plausibly enough, that it is for neo-logicists to say what conditions a logic has to meet if it is to be suitable for their purposes. But we may take it that it will suffice for those purposes if some sufficiently deductively potent second-order logic may be regarded as importing no significant theoretical hostages not, in effect, already offered by more basic systems of logic, par excellence, by first-order logic. For those willing to countenance the notion, Paul Boghossian's (1996 and 2003) idea of *epistemic analyticity* seems to be in the right territory, where a statement or principle is epistemically analytic just if its truth, or validity, may be recognised purely on the basis of an understanding of the concepts it configures.

[2] Here I will stick to "neo-logicism". A thorough overview of the distinctive claims of neo-logicism and the challenges confronting it is offered in Hale and Wright (2001).

syntactically inexplicit set theory. Since the reducibility of arithmetic, and analysis, to set theory, is no news and is arguably of at most minimal epistemological or metaphysical significance, it is a presupposition of the interest of the neo-logicist project that there should be some well-grounded alternative way of thinking about the higher-order logic required other than the Quinean way—some way consonant with core idea of logicism, that logical knowledge and at least basic mathematical knowledge are, in some important sense, of a single logico-epistemic kind.

This paper distinguishes four related though separable concerns about higher-order logic in general and about the specific demands placed on it by the neo-logicist project in particular. The paper has two principal objectives. The first is critically to review the responses to some of these concerns in recent work of the late Bob Hale, Richard Kimberly Heck and Crispin Wright. It is argued that some successes have been scored by these authors. The second objective, however, is to make a case that the most serious objection to the use made of higher-order logic in the execution of the neo-Fregean project not only remains unaddressed, but has not been sufficiently clearly understood.

This objection concerns the unavoidable recourse[3] that the neo-Fregeans make at crucial points in their constructions to *impredicative* higher-order definition and quantification. Of course, the legitimacy of recourse to impredicative quantifications and definitions is an old bone of contention among philosophers and mathematicians interested in the foundations of mathematics. Many theorists, both of Frege's day and more recent,[4] have jibbed at it. But their complaints have usually been fuelled by the role played by impredicative definitions in the generation of the paradoxes or, in some cases, by an implicit constructivism about mathematical *objects*. The concern to be developed here is different, and specific to impredicative higher-order *predication*. If it has been clearly perceived before, I do not know that it has been clearly stated.

The thrust of the objection is this: if no fully predicative second-order logic can underwrite a neo-Fregean proof of the Dedekind-Peano axioms or the construction of a completely ordered field of real numbers, then the neo-logicist project for arithmetic and analysis remains potentially blocked on the misgiving to be articulated below— a misgiving not about whether the impredicative proof-theory needed should properly count as 'logic', nor about whether impredicativity plays a villain's part in the spawning of paradox, nor yet about whether it is at odds with a desirable constructivist metaphysics of

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[3] Hume's Principle (see below), when set in a predicative second-order logic and accompanied by Frege's own definitions of the arithmetical primitives does not entail the Peano axioms, in particular the axiom that every number has a successor. See Linnebo (2004). John Burgess has observed that recourse to alternative, somewhat unintuitive definitions reinstates the entailment (Burgess (2005) at 155 ff.). Sean Walsh (2012) has shown that first order Peano arithmetic cannot be interpreted in predicative second order logic with Hume's Principle. See Linnebo (2016) for an overview of the technical limitations for the neo-logicist programme imposed by a restriction to predicative logic.

[4] *Loci classici* are Poincaré (1906), Russell (1908) and Weyl (1918); more recent kindred spirits are Parsons (2007) and Goldfarb (1988). Parsons (2017) evidences how constructivism was often not the central issue in play in the early debates about impredicativity. Dummett (1991, chapter 17) can be read as laying the blame for the intrusion of the "serpent of inconsistency" into *Grundgesetze* squarely at the door of the unrestricted impredicative comprehension licensed by Frege's substitution rules, a diagnosis whose relations to Dummett's other diagnosis, that Frege erred by overlooking the phenomenon of the *indefinite extensibility* of fundamental mathematical domains, is sympathetically explored in Wright (2019). (The debates on this point should be sharply distinguished from the objections raised by Dummett (1998) to the *first-order* impredicative quantification involved in Hume's Principle itself, whereby the numerical operator gets to bind predicates themselves containing that operator, and to the putatively legitimating counterarguments offered in Wright (1998) and (1998a).)

mathematical objects, but about whether it so much as makes intelligible sense at crucial points in the key constructions.

As it will be developed here, this objection seems to me an extremely challenging one. But there is a direction, prefigured in recent work of Øystein Linnebo, for meeting it head-on. I will conclude by outlining it.

§1 *First concern— the Range problem*

As will be familiar to most readers, what has come to be known as "Frege's Theorem" is the higher-order logical derivation, first outlined explicitly in the modern literature by Wright (1983), of the Dedekind-Peano axioms for arithmetic from what is now known as Hume's Principle (HP): that the number of Fs is identical to the number of Gs just if the Fs and the Gs admit of a one-one correspondence. Much discussion of the result has understandably focused on the definitional or logical credentials of HP. However, the significance, metaphysical or epistemological, which one will attribute to FT is going to vary not just with one's view of the status, metaphysical or epistemological, of Hume's Principle but also with one's opinion of the inferential machinery— traditionally, classical impredicative second-order logic with unrestricted comprehension— whereby the result is proved.

Classical impredicative second-order logic with unrestricted comprehension was, in effect, Frege's invention,[5] and he seems never to have entertained any doubt that his brainchild was fit for his logicist purpose. It's notable that in *Begriffschrift* he makes nothing of the distinction between quantifying into positions occupied by what he called *eigennamen* (singular terms) in a sentence and quantification into predicate position or, more generally, quantification into open sentences—into what remains of a sentence when one or more occurrences of singular terms are removed. Not only does he seem to have conceived of both alike as perfectly legitimate forms of generalisation, each properly belonging to logic. More: he seems to have conceived of quantification *as such* as an operation of pure logic and, throughout his career, to have recognised no important distinction between first-order, second-order and higher-order quantification in general.

This way of thinking is actually quite natural. Why did it become controversial? How did it come to be so widely believed that in his treatment of quantification Frege overlooked a major distinction?

One short answer is: "Quine".[6] Writing in a philosophical milieu when 'ontic parsimony' seemed a more urgent *desideratum* than perhaps it does now, Quine was preoccupied with possibilities of masked ontological commitments and, conversely, of merely apparent ontological commitments. His proposal, famously, was: regiment a theory using the syntax of individuation, predication and quantification and then see what entities you need to regard as lying in the range of the bound variables of the theory if it is to rank as true. You are committed, as a theorist, to what you need to quantify over in so formulating your theory. (As to the question of how we are supposed to recognise the

[5] As Richard Kimberly Heck has observed (Heck 2018), this way of characterising Frege's achievement is not quite right. There are no comprehension axioms in the systems of *Begriffschrift* and *Grundgesetze.* Rather, Frege has a rule of substitution: where F is a predicate of whatever degree and [. . .F . . .] is a theorem, one may—subject to standard restrictions—infer the corresponding [. . . ϕ . . .] resulting from replacing all occurrences of F in [. . . F . . .] by any open sentence, ϕ, of the same degree. As Heck notes, the substitution rule is equivalent to full impredicative comprehension. The proof of Frege's Theorem in Wright (1983) proceeds via the substitution rule (not there explicitly formulated.)

[6] At least, it was Quine who was largely responsible for persuading several generations of *philosophers* that the distinction between first and higher-order quantification marked a crucial ontological and epistemological watershed.

adequacy—or inadequacy—of such a regimentation, Quine had, of course, relatively little to say.)

Quine's cardinal thought about quantification was thus that it is intimately tied to—indeed, is the canonical expression of—ontological commitment. This is not *as such* at odds with the Fregean idea of quantification as a level-neutral uniform logical operation. But once you fall in with it, first- and second-order quantification do suddenly emerge as standing on a very different footing. First-order quantification quantifies over objects. No one seriously doubts the existence of objects. By contrast, higher-order quantification is naturally taken to demand a domain of universals, or properties, or concepts. And of such entities Quine canvassed an influential mistrust: a mistrust based, initially, on their mere abstractness—though he himself later, under pressure of the perceived needs of science, swallowed something of his antipathy to the abstract—but also on the ground that they seem to lack clear criteria of identity: a clear basis on which they may be identified and distinguished among themselves. It was the latter consideration which first led Quine to propose that the range of the variables in higher-order logic might as well be taken to be *sets*—abstract entities no doubt, but ones with a clear criterion of identity given by the axiom of extensionality—and then eventually to slide into a view in which "higher-order logic" became, in effect, a misnomer, unless, at any rate, one regards set theory as logic. By 1970 he had come to his well-known view:

> Followers of Hilbert have continued to quantify predicate letters, obtaining what they call higher-order predicate calculus. The values of these variables are in effect sets; and this way of presenting set theory gives it a deceptive resemblance to logic .... set theory's staggering existential assumptions are cunningly hidden now in the tacit shift from schematic predicate letters to quantifiable set variables.[7]

Those remarks occur in Quine's chapter, "The Scope of Logic" in the sub-section famously entitled: "Set Theory in Sheep's Clothing"! By the end of that chapter, Quine has persuaded himself, and probably most of his readers too, that Frege and others such as Russell and Hilbert who followed him in allowing higher-order quantification at all, simply overlooked the implicit transition from logic properly so regarded—the theory of the valid patterns of inference sustained by the formal characteristics of thoughts expressible using singular reference, predication, quantification and identity—to set theory which, to the contrary, in Quine's view, is properly regarded as a specialised, though highly general and fruitful, branch of mathematics.

Quine's disparagement of higher-order logic as crypto-set theory has of course met with resistance over the years, notably—to cite two rather different examples—in various of Bob Hale's writings about the issues,[8] and earlier in those of George Boolos' papers that develop the idea that at least as far as monadic higher-order generalisation is concerned, quantification can be treated as plural quantification over individuals.[9] Both Hale and Boolos, though, retain the Quinean assumption that quantification, of whatever order,

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[7] Quine (1970), p. 68.

[8] Hale (2012), (2013) and (2015).

[9] See especially the articles, "To Be is to be the Value of a Variable (or to be Some Values of Some Variables" and "Nominalist Platonism" reprinted as chapters 4 and 5 in part 1 of Boolos (1998). Boolos was, of course, well aware that there is no obvious way of extending this suggestion to the interpretation of quantification into the positions occupied by predicates of higher degree. His goal was not to give a thoroughgoing non- set theoretic interpretation of higher order quantification but rather to provide a means for interpreting certain generalisations about 'all sets' that strike us as intuitive and true. An heroic effort at polyadic extension of Boolos' idea is nevertheless made by Rayo and Yablo (2002). For pertinent criticisms, see Rossberg (2015).

essentially involves a *range of entities* about which it gives us the resources to generalise in various ways, and to which its use commits us. This view of the matter has long been explanatory orthodoxy.[10]

It is, however, not obvious that it is compulsory. In a (perhaps deservedly) somewhat neglected paper published a while ago,[11] I argued that the conception of quantification as *essentially* range-of-entities-involving is potentially a serious misunderstanding — or at least an avoidable interpretation —of what quantification fundamentally is. That paper proposed, by contrast, a *Neutralist* conception of higher-order quantification—and indeed of quantification more generally—as expressed by the following principle:

> (*Neutrality*): Quantification into the position occupied by a particular type of syntactic constituent in a statement of a particular logical form cannot generate an ontological commitment not already associated with the occurrence of that type of constituent in a true statement of that form.[12]

According to Neutrality, the ontological commitments of quantified statements go no further than those of their instances, whatever the commitments of the latter may be. So the Neutralist view is open to someone who thinks that simple predications of "red", e.g., *do* semantically entrain a universal of redness, or a set, or a trope, or some other kind of entity distinctively associated with that predicate as its semantic value. If we take the view that the semantic values of the predicates in a particular language are sets, Neutrality is consistent with regarding the standard semantics for a higher-order logic for that language as part of the theory of its meaning. But a mixed attitude — that of holding that predication is ontologically neutral while higher order quantification, like all quantification, is essentially range-of-entities involving — is pre-empted. Arguably this was indeed Quine's own view: that while mere predication *is* free of any distinctive ontological commitment, quantification into predicate position must commit a thinker to an *extra* species of entity—in the best case, sets; in the worst, attributes or universals. This, according to Neutrality,[13] is an incoherent stance.[14]

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[10] For instance in the entry under "Quantifier" in Blackburn (1994) his *Oxford Dictionary of Philosophy*, Simon Blackburn writes:

> Informally, a quantifier is an expression that reports a quantity of times that a predicate is satisfied in some class of <u>things</u> , i.e. in a 'domain' [my emphasis, p. 313],

while the corresponding entry at p. 338 in Antony Flew's and Stephen Priest's *Dictionary of Philosophy* Flew & Priest (2002) observes that

> The truth or falsity of a quantified statement.... cannot be assessed unless one knows what totality of objects is under discussion, or where the values of the variables may come from. (p. 338)

[11] Wright (2007)

[12] Unbeknownst to me, the basic idea of Neutrality was not new when I hit upon it but arguably had been anticipated in the writings of both Stanislaw Lesniewski and Arthur Prior. For details, see Peter Simons (1997).

[13] Neutralism is thus in some respects akin in spirit to a substitutional conception of quantification. A crucial difference, though, is that neutralism need not be conceived as a license to quantify into whatever grammatical categories — including prepositions, adverbs, verb inflections, and whatever else, —happen to feature in a particular language, but is best conceived as a proposal concerning quantification into elements of logical form, the constituents of structured (Fregean) *thoughts*.

[14] Cf. Rayo and Yablo (2001). The objection is prefigured in Wright (1983) at p. 133. A referee for this journal has pressed on me the observation that it would be open to Quine to defend against the objection in the text by accepting only the following emasculated version of Neutrality:

What, though, if we *don't* take the view that a semantics of predication should associate a distinctive kind of entity with each meaningful predicate as its semantic value, holding instead, with Quine, that predication is free of any distinctive ontological commitment. In that case Neutrality gives us a choice. We can hold that quantification as such does essentially implicate ontological commitment, and hence that quantification into predicates and open sentences generally misrepresents the ontological innocence of predication and is thereby illicit. Or we can take the view that I set against the range-of-entities-involving conception: viz. that quantification through expressions of any syntactic kind — that is, any expressions associated with a specific type of semantic contribution, partially determinative of the logical form of the proposition expressed by a sentence in which they occur—should be viewed as essentially a device for generalising exactly that: a device for *generalisation of semantic contribution*. Given any logico-syntactic category of which an instance, *s*, can significantly occur in a context of the form, [...*s*...], quantification through the place occupied by '*s*' is to be thought of as a function which takes us from [...*s*...], conceived purely as a structured content, to another content whose truth conditions are given as satisfied just by a certain kind (and quantity) of distribution of truth values among contents differing from […*s*…] only through variation in what takes the place in their structure occupied by '*s*'. A quantifier, of whatever order, is a function which takes us from a statement of a particular logical form to another statement which is true just in case some range of (possible) contents—a range whose extent is fixed by the quantifier in question—which share that same logical form are true.

According to Neutralism, then, what if any entities are associated with higher-order quantification turns on one's semantics of predication. The commitment one undertakes in affirming, e.g., a second order universal generalisation is to the truth of all of a certain range of thoughts sharing a certain logical form—perhaps to all possible such thoughts, perhaps to all such thoughts expressible in a given, open-ended language, perhaps to some more narrowly contextually determined range. If, with the nominalist, we take the view that no thought of that structure makes any reference to, or otherwise draws on, an entity such as a property, or universal, or set, the result is a nominalism-friendly conception of higher-order generality; if, by contrast, with Hale, we take it that the essence of predication is the ascription of a property, intensionally conceived, then neutrality will deliver the conception of higher-order generality that Quine reacted against. But note in that case that first-order logic too, insofar as it is a logic of schematic predication, will incur the same ontological commitments. For Neutralism, there is no major watershed in this regard between first- and higher-order logics: second-order logic and indeed *n*th-order logic have, when their quantifiers are conceived along neutralist lines, and as far as their ontological commitments are concerned, exactly the same claim to be logic as first-order logic.

Above I cited Bob Hale as among the critics of Quine's views about second-order logic who nevertheless retain the Quinean assumption of quantification as essentially range-of-entities-involving. In fact, though, the interpretation of second-order logic that Hale favoured is exactly what Neutralism provides when adjoined to what is now standardly termed an *abundant* conception of properties. According to abundance, any well-defined predicate determines a property, that is, *a way things can be*. For any well-

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> Quantification into the position occupied by a particular type of syntactic constituent in a statement of a particular form, *if it is legitimate at all*, cannot generate an ontological commitment not already associated with the occurrence of that type of constituent in a true statement of that form.

This would enable Quine to regard predication *per se* as ontologically neutral, higher-order quantification as ontologically loaded and to transmit this conjunction into a denial that higher-order quantification is indeed legitimate. I must forbear to pursue this dialectic further here.

defined predicate is associated with a satisfaction condition, meeting which is, naturally, one way for a thing, or things, to be. But conversely, also, Hale held that properties extend no further than the possibilities for well-defined predication: a property just is a way which some possible well-defined predicate might be used to say how a thing, or things, are. In Hale's view, save where contextual restrictions apply, the range of the quantifiers in second-order logic should encompass all properties so conceived. Since any well-defined predicate must be finitely defined, it would seem to follow that the domain of Hale-properties will represent a considerable contraction by comparison with the domain of the predicate variables in second-order logic under its standard set-theoretic semantic interpretation, which encompasses the full power set of the first-order domain and hence, when the latter is infinite, includes an uncountable population of entities that, at least in any particular countable symbolism, allow of no finite specification.[15] Still, it remains the case, so far as I can see, that there is, in one way, very little difference *in spirit* between Hale's view, so motivated, and the interpretation of second-order logic that Neutralism issues on a nominalist view of predication, whereon even the metaphysically lightweight properties involved in the abundant conception are disdained. Sure, such a neutralist-nominalism will balk at the metaphysics of abundance. But on both views, the commitment one undertakes in affirming a higher-order quantified statement, and hence what has to be established in order to justify one in doing so, will be that there exists a suitable range of intelligible true thoughts to bear witness to the truth of the former, *modulo* the extent of the generality involved in the quantification concerned.

§2 Second concern — the Domain problem

Neutralism offers a head-on response to Quine's ontological misgivings. We will not here further explore whether it can in the end provide a fully satisfactory conception of higher-order quantification in general, though below we will confront some issues about whether it can accommodate certain of the specific demands placed on higher order logic by the neo-Fregean programme. In any case concern about the nature of the entities putatively quantified over—what we have termed the *Range problem*—is merely one of several misgivings that classical second-order logic with full impredicative comprehension has tended to provoke. Richard Kimberly Heck characterises a second as follows:

> In the case of second-order logic, however, there is another and more fundamental problem with which we must contend: we must explain the second-order quantifiers. Absent such an explanation, we do not so much as understand second-order languages . . . Now, to understand the second-order universal quantifier, one must understand what it means to say that *all* concepts are thus and so. But to understand that sort of claim, so it is often argued, one must have a conception of what the second-order domain comprises. One must, in particular, have a conception of (something essentially equivalent to) the full power set of the first-order domain, and many arguments have been offered that purport to show we simply do not have a definite conception of [that].[16]

[15] One might have expected that so restricting the domain of the higher-order variables would impact on the standing of the metatheoretic properties—categoricity, deductive incompleteness, non-compactness and failure of the Downward Lowenheim-Skolem theorem—that are usually viewed as marking an important watershed between second- and first-order logic. As Hale (2015) notes, however, this is not so: these results can go through as before.

[16] Heck (2020) 46-7

Heck wants to claim that their *Arché Logic*, of which more shortly,[17] finesses this second concern—call it the *Domain problem*—by dispensing with second-order quantification altogether. As they realise, however, the point cannot be that quick, since Arché logic has its own resources for the expression of higher-order generality and there has therefore to be a substantive question whether understanding the generality it does express presupposes the allegedly problematic conception of the full power set of the first-order domain. Naturally, that conception is not required on Hale's interpretation of the range of the second-order quantifiers. But it might be contended that an analogue of the problem carries over to Hale, only now the problematic conception is that of *all possible intelligible n-adic predicates (open sentences)*. True, the foreseeable grounds for suspicion of this notion are likely to differ from those that some[18] have found with the idea of the classical power set of an infinite domain. The latter have to do with the implicit idea of an arbitrary infinite sequence of operations: the unspecifiability of the putative results of such an operation in uncountably many cases, and the implicit reliance on the idea of the completed infinite involved in the very idea that there could be such a thing as 'the result'. Concern about *all possible intelligible n-adic predicates*, by contrast, is more likely to focus on misgivings whether there can be any determinate such totality— since one would expect to be able to "diagonalise out' of any well-specified collection of such expressions and it is therefore questionable whether it is possible to possess any definite, stable conception of the domain of second-order logic's proprietary quantifiers as Hale conceives of it, at least when the quantification concerned is intended to be unrestricted.

Heck has, as it seems to me, a strong response to a critic who claims that Arché logic sweeps the domain problem under the carpet rather than serving to address it. The example under discussion is the antecedent of their proposed introduction rule for the ancestral; specifically, writing ϕ^*ab for '*a* bears the ancestral of the relation ϕ to *b*', the rule:

$$(\text{*-intro}) \qquad \frac{[\forall x(\phi ax \rightarrow Fx) \;\&\; \forall x\forall y(Fx \;\&\; \phi xy \rightarrow Fy)] \rightarrow Fb}{\phi^*ab}$$

In Arché logic, this is a first-order *schema*; there is no binding 'F'. Nevertheless, Heck imagines a critic contending, our understanding of it

> . . . essentially involves just such a conception [of the full power set of the domain]. How else are we to understand . . . the premise of the rule . . . except as involving a tacit initial second-order quantifier? Does it not say, explicit quantifier or no, that all concepts F are thus-and-so? Does understanding that claim not require the disputed conception of the power set?

And Heck replies:

> No, it does not. A better reading would be: a concept that is thus-and-so is so-and-thus. What understanding this claim requires is not a capacity to conceive of *all* concepts but simply the capacity to conceive of *a* concept: to conceive of an arbitrary concept, if you like. The contrast here is entirely parallel to that between arithmetical claims like $x + y = y + x$ involving only free variables and claims involving explicit quantification over all natural

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[17] I will explain Arché logic in more detail below. But basically, Arché Logic is a regular first-order logic to which are added definitional resources such that, whenever Φ[… Fx….y] is an open sentence in exactly one free individual variable, y, that holds good for any substituend for 'F', — for instance "Fy—>Fy", "~Fy —> (∃x)~Fx", .. , etc. — the introduction is permitted of a new monadic first-order predicate, "Ay", stipulated to be equivalent to Φ[… Fx….y]. This, as Heck shows, allows the simulation in a first-order setting of enough of the effect of monadic second-order quantification to facilitate a proof of Frege's Theorem. Further details to follow in §5.

[18] See e.g. Wright (1985).

> numbers. Hilbert famously argued that our understanding of claims of the former sort involves no conception of the totality of all natural numbers, whereas claims of the latter sort do, and that there is therefore a significant conceptual and epistemological difference between these cases. I am making a similar point about claims involving only free second-order variables as opposed to claims that quantify over concepts.[19]

In fact, however, Heck is here making two separable points at once: first, that there is a significant conceptual and epistemological distinction between having the notion of *an arbitrary* thing of a certain kind and having a notion of *all* things of that kind, in a sense involving some kind of conceptual representation of their totality; and second, that this distinction aligns with that between grasp of the meanings of statements, like the premise for *-intro, involving free variables, and grasp of the meanings of the statements that result from them by binding those variables with quantifiers. My sympathies are entirely with Heck and Hilbert on the first point (though more needs to be said about it.) But the second seems merely legislative. Granting the distinction in question, why insist that its second component is implicated in quantification *per se*— especially if that component is regarded as problematic in the crucial second-order case? Why should not both free-variable statements *and* universally quantified statements serve to express just schematic generality? Hale can say that that is precisely how universal quantification over abundant properties is to be understood.[20] And on the neutralist understanding, a schematic interpretation of quantification is provided for from the start.

§3 Third Concern — Neutralism and Comprehension

That is all I want to say here about the Domain and Range problems. So far, so good, it may be thought for the prospects for a neo-logicism -friendly understanding of the logic it requires. But now the horizon darkens. The key question is whether it is safe to assume that an interpretation of second-order logic of the kind sanctioned by nominalist-neutralism, or as suggested by Hale, has no implications for the proof-theoretic power of the system, in particular, whether such an interpretation poses no barriers to the derivation of Frege's Theorem and the reconstruction of our judgements of finite cardinal number in general.

Wright (2007)[21] already noted that Neutralism will struggle to underwrite the classical second order comprehension axioms in any case where the domain of individuals is taken, for whatever reason, to include elements which do not allow of any individuative specification. Classical comprehension has, for example, this instance:

$$(\forall x)(\exists X)(\forall y)(Xy \leftrightarrow y=x)$$

- intuitively, that for anything, there is a property which anything has just in case it is identical with that thing. This chimes with Neutralism's required connection between a second-order quantification and the intelligibility of the range of thoughts that constitute its instances only if each object in the first-order domain in question—in the range of 'x' —is itself a possible object of intelligible singular thought. With uncountably infinite domains

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[19] Heck (2020) p. 48

[20] To avoid misunderstanding: I am not of course claiming that a universally quantified sentence has the same sense as its unquantified, schematically general counterpart. There is, naturally, an asymmetry under negation: $\sim(\forall x)\phi x$ diverges in truth-conditions from $\sim\phi x$, when the latter is understood schematically generally. But the question is whether this difference enforces a reading of the former which implicates a conception of the *totality* of elements in the range of the quantifier. It seems to me that it obviously does not, but requires at most only that it be determinate of any particular object whether it falls within that range.

[21] Pp. 165-6.

that is controversial. Only an infinite notation could provide the means *canonically*[22] to distinguish each classical real number, for example, from every other in the way that the standard decimal notation provides the means canonically to distinguish among the naturals. So, on plausible assumptions, no finite mind can think individuative thoughts of each and every real. Yet the instance of comprehension cited implies that to each of them corresponds a distinctive property.

More generally, consider, for example, *Grundgesetze* theorem §118, which in the notation of Wright (1983) is

$$Ny{:}(y = a) = Ny{:}(y = b)$$

Here it is essential to understanding the point of the theorem to take '*a*' and '*b*' as ranging schematically over any objects whatever, and not merely over objects for which we are able to grasp some individuative specification. Any restriction of the range of parameters like the displayed occurrences of '*a*' and '*b*' to objects available to intelligible singular thought will not merely undercut the generality of theorem 118, but will pose impossible obstacles to the construction the theory of the real numbers, at least as classically conceived.

The difficulty rears again when one considers the prospects for a construction of Complex Analysis in the context of Neutralism. Consider, for example, the proof that $Nx{:}(x^2 = -1) = 2$. Since we necessarily lack any mutually individuative conceptions of the two complex square roots of -1, Neutralism has, it appears, no satisfactory account to offer of the meaning of e.g. $(\exists x)(x^2 = -1)$, nor therefore of $(\exists x)(\exists y)[(x^2 = -1 \;\&\; y^2 = -1 \;\&\; x \neq y \;\&\; (\forall z)(z^2 = -1 \rightarrow z = x \text{ or } z = y)]$.[23,]

Thus the tie effected by Neutralism between first-order quantification and intelligible singular thought may be a potential a road-block in the way of any neo-logicist ambition to go beyond arithmetic and provide abstractionist foundations for real and complex analysis also.

§4 Fourth concern — the Impredicativity problem.

However there is a deeper worry about Comprehension nearby: a worry whether *any* significant part of the neo-logicist enterprise, including the derivation of Frege's Theorem, can be accomplished if the underlying logic is fully compliant with nominalist-neutralist constraints. The supporter of second-order logic has, after all, *two* distinctive obligations associated with second-order generalisation. The first is to explain the *meanings* of the quantifiers — and it is to this task that both the nominalist-neutralist proposal and Hale's abundant-property suggestion are intended to contribute. But the second is to explain the *scope* of the quantifiers— to explain what range of open sentences in one or more free objectual variables are to be eligible for second-order binding. Frege's answer, in keeping with his comprehensive conception of the scope of objectual quantification, was: "All of

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[22] Where a canonical specification of an object is one grasp of which *eo ipso* imports understanding of which object is concerned, understanding of its identity. But it is an issue whether intelligible singular thought understood as for the purposes of Neutralism, should demand *canonical* specification of the objects concerned. Cannot we have particular thoughts about any real number, even if we can't simultaneously have terms in the language for all of them? Suppose you have a spinner with a pointer; you give the spinner a random jolt and it runs around, stopping to form some angle theta from where it started. Claim: theta can be any positive real—more specifically, letting the 0 point be the vertical orientation, any positive real in the interval $\{-\pi/2, \pi/2\}$; and I can have *de re* thoughts about it, as the resulting angle of that spin. (This was urged on me by Chris Scambler.)

[23] — an observation due to Heck.

them" —*any* well-formed such open sentence is so eligible. But the availability of that answer depends on how one responds to the first obligation. The salient worry, of course, is *impredicativity*: specifically, it is whether nominalist-neutralism, or Hale's abundantism, can make any sense of impredicative higher-order quantification in general. Wright (2007) anticipates the concern but I regard the tone of the discussion there as much too sanguine. Reflection on harmless cases like his "Federer has all the qualities of a great champion" — where that very quality is, naturally, one that someone who has *all* the qualities of a great champion had better have! — masks the difficulties which nominalist-neutralism and, I contend, Hale's proposal too, here encounter.

The vexed and controversial character of the debates about impredicative specification ever since Poincaré and Russell first expressed concerns is evidence that it is challenging to frame the question of its legitimacy precisely. In fact there are a number of distinct concerns about impredicativity. One is occasioned by the need for self-instantiation of impredicatively characterised functions and concepts in the standard ways of generating the paradoxes on which Russell and his interlocutors in the earliest debates about impredicativity were focussed. Another concern, famously dismissed by Ramsey and Gödel, is the apparent tension between allowing impredicative means of specifying the entities dealt with by a mathematical theory with any broadly constructivist conception of that theory's subject matter. In the present context, though, I would urge the reader to keep in focus that we are not primarily concerned with either of those questions now nor with any version of the *general* question whether, or under what conditions, one may determinately, or legitimately, single out an entity of any kind by quantification over the elements of a set or other kind of grouping of which it is a member. Our question is, rather, one which has received but minimal, if any, specific attention in the literature: the question of what it takes to endow a higher-order predicate with a determinate *sense*.

The looming problem is essentially one of semantic *grounding*. Let us pause to think it through. A higher-order predicate is relevantly semantically grounded only if its satisfaction conditions can be finitely, and non-circularly explained by reference to the satisfaction-conditions of the predicates in its range of quantification. Assume that such a predicate has determinate satisfaction-conditions — a determinate sense —only if so grounded and consider accordingly any base class of well-understood first-level[24] (for simplicity's sake, monadic) predicates containing no higher-order quantifiers. Let "Fy" be one of these *basic* predicates. Now consider any first level predicate "$QX(Xy)$" formed by some mode of higher-order quantification, Q, into "Fy", and let it be understood that the quantification is impredicative, so that along with all the basic predicates, "$QX(Xy)$" itself falls within its own range. Any predicate in good order must possess determinate, finitely intelligible satisfaction-conditions. So, what is the condition on an object, a, if it is to satisfy "$QX(Xy)$"? Well, obviously, it is that a should satisfy Q-many of the predicates[25] in the range of the quantifier. But is that a determinate requirement? There are two cases.

Suppose (Case1) that the truth-value of any instance, "$QX(Xa)$", is always determined by a's satisfaction, or violation, of some appropriate number of basic predicates. Then the impredicativity of the quantifier is semantically idle. We could generate something with the same truth-conditions as "$QX(Xy)$" by means of a predicative quantification instead. This is how it is with "Federer has all the qualities of a great champion": while the Swiss tennis star had naturally better have that quality too if the

[24] Lest there be any confusion: "first-level" here indicates a predicate of objects. Contrast "first-order", which indicates, as is usual, a predicate free of higher-order quantifiers.

[25] To avoid complication, I am here bracketing questions about possible dependence-relationships among the predicates in question. Doing so does not affect the relevant point to follow.

predication is to be true of him, he can presumably have it just in virtue of satisfying other predicates which are free of higher-order quantifiers. (In effect, an axiom of reducibility, in the sense of *Principia Mathematica*, holds in Case 1.)

Thus if the impredicativity of "*QX*(*Xx*)" is not to be semantically idle, but is to make some material difference to its satisfaction-conditions, — if it is to be, as we may say, *actively impredicative*, —then (Case 2) what makes "*QX*(*Xa*)" true must turn on *a*'s satisfaction of Q-many predicates in a range that includes some non-basic predicates, involving higher-order quantifiers. But obviously we make no progress in explaining how "*QX*(*Xx*)" can indeed have determinate satisfaction-conditions if we simply say that it is itself included in its range and that this makes a material difference to whether any particular object satisfies it. So it seems that a model of how an actively impredicative "*QX*(*Xx*)" can have determinate satisfaction-conditions must assign a range to its quantifier that both goes beyond the basic predicates and includes predicates *distinct* from "*QX*(*Xx*)". Let's call these *secondary* predicates for "*QX*(*Xx*)".

Now the problem is evident. These secondary predicates must also involve actively impredicative higher-order quantification. Otherwise, the real contribution made by *a*'s satisfaction of a secondary predicate to its satisfaction of "*QX*(*Xx*)" will be delivered by its satisfaction of appropriately many *basic* predicates instead, so that the situation will revert to Case 1 by proxy, as it were. So a model that explains how an actively impredicative "*QX*(*Xx*)" can have determinate satisfaction conditions must, it seems, attempt to do so by including in the range of its quantifier some *other* first-level predicate that is actively impredicative but nevertheless possesses determinate satisfaction conditions. But it was of exactly that possibility of which an explanatory model was demanded in the first place! We are spinning our wheels.

In summary: any higher order predicate "*QX*(*Xx*)" has determinate satisfaction-conditions only to the extent that it is a determinate matter what it is for an individual to satisfy Q-many of the predicates in its range, and hence only if it is determinate both what that range is and that the predicates it includes *themselves* have determinate satisfaction-conditions. The challenge is therefore to provide a model of how, when "*QX*(*Xx*)" is actively impredicative, — so that its satisfaction-conditions cannot be explained in terms purely of the satisfaction of Q-many basic predicates, —a larger class of predicates might be identified over which it ranges and which may be independently credited with determinate satisfaction-conditions. Since such a larger class will, for the reasons given, be constrained to include further actively impredicative non-basic predicates, it will deliver what is wanted only if the problem of semantic grounding — the problem with which we started — has already been solved for *them*.

I hesitate to present this reasoning as a proof that actively impredicative first-level predicates *cannot* be semantically grounded and hence cannot have determinate satisfaction conditions. What it proves, rather — if it proves anything — is that there is no clear prospect of a model that satisfactorily explains how they can. But if the pessimistic conclusion which that may suggest is right, then that kills off any chance that a second-order logic sufficient for the proof of Frege's Theorem, let alone for an abstractionist recovery of analysis, can be vindicated by nominalist-neutralism, or by Hale's approach. Heck's conclusion, "that [the] 'neutralist' account of quantification will not serve the needs of the neo-Fregean program",[26] will be sustained. And a similarly negative verdict will apply to Hale's attempted reconstruction of the ontology of second-order logic in terms of abundant properties.

[26] Heck (2018), p. 152

To be sure, to draw that conclusion is not to conclude that neither nominalist-neutralism, nor Hale, can respectively offer a satisfactory interpretation — even maybe a uniquely superior interpretation — of higher-order quantification as a bona fide logical operation. That is a further question. What is in doubt, rather, is, specifically, whether Hale's or my proposals can underwrite the higher order logical systems of *Begriffschrift* and *Grundgesetze* and, indeed, whether either can underwrite any higher-order logic strong enough to subserve the proof of (an interestingly strong version of)[27] Frege's Theorem.

§5 *Does Heck's Arché Logic serve the neo-logicist better? One worry: Paradox*

Arché logic is essentially a natural generalisation of *ancestral logic*: the system one gets by augmenting first-order logic with an operator, *, that when applied to any expression for a dyadic relation, yields an expression denoting the ancestral of that relation. In Heck's formulation, the ancestral operator is controlled — Heck suggests, "defined" — by a pair of rules closely following Frege's original explicit definition. The introduction rule, as noted above, is:

(*-intro) $$\frac{[\forall x(\phi ax \rightarrow Fx) \,\&\, \forall x\forall y(Fx \,\&\, \phi xy \rightarrow Fy)] \rightarrow Fb}{\phi^{*}ab}$$ [28]

The elimination rule is, naturally, simply the converse:

(*-elim) $$\frac{\phi^{*}ab}{[\forall x(\phi ax \rightarrow Fx) \,\&\, \forall x\forall y(Fx \,\&\, \phi xy \rightarrow Fy)] \rightarrow Fb}$$

Heck observes that these two rules illustrate a more general pattern of *schematic definition*. In regular second-order logic, wherever $\phi x(Fx,y)$ is a formula with no free variables save F and y, we can define a new predicate, $A\phi y$ by universally binding *F*; thus

$$A\phi y = \forall F\phi x(Fx, y)$$

—a predicate which expresses, roughly the property that any object y has just if, for every F, y satisfies the mixed-level binary relation, ϕ, on itself and F. What the rules for the ancestral illustrate is how we can always get the effect of that resource, without resort to second-order quantification, by laying down a pair of schematic rules, thus:

(A-intro) $$\frac{\phi x(Fx,y)}{A\phi y}$$ [29]

; and (A-elim) $$\frac{A\phi y}{\phi x(Fx,y)}$$

The generality involved in rules of this pattern is, as Heck emphasises, nothing more exotic more than the schematic generality already involved in theorems of first-order logic like Fx —> Fx, ~(Fx & ~Fx), and so on. Arché logic is simply first-order logic augmented with this definitional resource. And—remarkably — it suffices, as Heck shows, for the proof of Frege's Theorem.

[27] The qualification is necessary because, naturally, the strength of the induction axiom will vary with the range of the higher order variables it contains. I will omit it in what follows.

[28] — where *F* must not occur unbound in any premise on which the thesis above the line depends.

[29] —cf. n 16.

At least, it does so provided we allow ourselves a rule of substitution analogous to that which Frege assumed for the system of *Grundgesetze*: in effect, that in applying any elimination rule of the illustrated kind, we may substitute for ϕ *any* formula of the appropriate form whose formation the language of the system allows, that is, any formula with a single free individual variable — including formulae which contain occurrences of the relevant defined predicate, $A_\phi y$, itself. Augmented with this rule of substitution, Arché logic mimics the proof-theoretic power of a system of second-order logic with just Π^1_1 – comprehension and no second-order existential quantifier. But that is enough, as Heck demonstrates, for the derivation of Frege's Theorem. Without the substitution rule, on the other hand, the derivation is blocked.

So, is Arché logic "epistemically innocent"? One train of thought that might occasion doubt is that, in conjunction with a first-order schematic version of *Grundgesetze* Law V, it suffices for Russell's paradox. It is very often supposed that the paradox showed that Law V is inconsistent, — the 'Bad Company' problem for abstraction principles is often presented in ways that assume so. But as I have argued elsewhere, all that is initially clear is that the paradox is the product of the *interaction* between Frege's axiom and his underlying logic.[30] It is a philosophical question which, if either, deserves the Lion's Share of the blame.[31]

One relevant fact is that when Law V itself is adjoined to a second-order logic admitting only predicative comprehension, no paradox ensues.[32] Another is that a first-order *schematic* version of Law V is perfectly consistent in the setting of classical first-order logic. Not so, however, when the underlying logic is Arché logic. Here is one way of obtaining a version of Russell's antinomy. First assume Law V in schematic form:

(V) $\{x{:}Fx\} = \{x{:}Gx\} \longleftrightarrow (\forall x)(Fx \longleftrightarrow Gx)$

Substitution of *F* for *G*, *modus ponens* on the right-to-left ingredient conditional and existential generalisation gives us *schematic naive comprehension*:

$$(\exists x)(x = \{x{:}Fx\})$$

Next we apply the schematic rules A-intro and A-elim above to the following open sentence,

$$y = \{x{:}Fx\} \longrightarrow {\sim}Fy$$

Thus:

R-intro $\dfrac{y = \{x{:}Fx\} \longrightarrow {\sim}Fy}{Ry}$ [33]

R-elim $\dfrac{Ry}{y = \{x{:}Fx\} \longrightarrow {\sim}Fy}$

By naïve comprehension, we have that *R* is associated with an extension:

$$(\exists x)(x = \{x{:}Rx\})$$

— call this object ***r***. Suppose *Rr*. Then by *R*-elim

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[30] Wright (2019).

[31] Something of a classic exchange on the question is provided by Boolos (1993) and Dummett (1994).

[32] This was first noted by Heck (1996)

[33] —cf. n 16.

$$\boldsymbol{r} = \{x{:}Rx\} \longrightarrow \sim R\boldsymbol{r}$$

Since we have the antecedent of the latter, we obtain ~*R**r* — contradiction. So discharging the assumption of *R**r* by reductio, we obtain ~*R**r* depending just on schematic V as premise. That suffices for the premise for an *R*-intro step meeting the necessary restrictions. Whence *R**r*.

So, the critic may contend, its relatively spare expressive and proof-theoretic resources notwithstanding, Arché logic still suffices for paradox when annexed to a principle that is merely a pale shadow of *Grundgesetze* Law V— a principle which in *regular* first-order logic is perfectly harmless. The source of the problem is evident: it is the substitution rule which allows an arbitrary predicate introduced by means of the A-schemata to go on to occur within the scope of '*F*' in the associated rules. Syntactically viewed, this is the sparest, most modest form of impredicativity. But the manoeuvre it allows *suffices* to generate paradox when aligned with — so the critic may suggest —an otherwise innocent principle, as we have just seen, — and, crucially, is *necessary* for the proof of arithmetical induction by Arché-logical means which Heck provides. How then can it be trusted in the latter context?

Though they do not address it to exactly this question, Heck makes an observation *ob iter* which may be thought to answer it. It is that if we add '∈' — set-membership — as a primitive to a regular first-order language with schematic V as an axiom, and set it as controlled by the principle they call *∈-lite*:

$$(y)\ (y \in \{x{:}Fx\} \longleftrightarrow Fy)$$

then, taking Fx as x ∉ x and y as {x: x ∉ x} — as naïve comprehension permits —we immediately have the paradox again using only regular first-order logic. So —one might argue — since *∈-lite* is absolutely intuitive as a characterisation of set-membership[34] — there is cause to think that schematic V is a bad principle anyway, and hence that its generation of paradox in Arché logic need not reflect adversely on the latter. However selective finger-pointing at the sources of a paradox is a slippery business. What this last paradox shows is that the intuitive notion of set that combines schematic V, hence naïve comprehension, with the conception of set-membership embodied in *∈-lite* is *per se* an incoherent notion. But then we have once again, not a single suspect but two: to allow that every open sentence in one objectual variable determines a set is not yet a commitment to the idea that any satisfier of that open sentence is a member of that set. The paradox in Arché logic does not exploit any particular view on that, or on how membership should be characterised in general, but presupposes only the admissibility of the general idea of a set of F-elements not itself being F. Heck's observation shows that if we are committed to *∈-lite*, we have no reason to look askance at Arché logic as a source of the latter paradox. But it is also consistent with the stance that schematic V is innocent, but vulnerable to mishap both when set in Arché logic and when conjoined with ∈-lite.

As mentioned, I argue[35] that the original paradox in the system of *Grundgesetze* should, at least in the first instance, be viewed not as a *reductio* of law V but as a demonstration of the incoherence of Frege's unthinking combination of the notion of logical object incorporated in law V—whereby any well-formed open sentence has an associated object, an extension or more generally a value-range — with the unrestricted predicate comprehension resources made available by his substitution rule which allows any open sentence, of whatever degree of quantificational complexity, to be a legitimate instance of higher-order quantification. What we have with schematic V and Arché logic is

[34] — "The definition of membership one would really like to give". Heck (2020, p. 30.

[35] In Wright (2019).

just a pared-down tension of essentially the same species: Heck's substitution rule, whereby any predicate introduced by instances of the A-schemata may be taken in turn as an admissible substituend for their contained occurrences of predicate variables will not marry with the concept of logical object encoded in schematic V. Indeed there is a case for saying that, rather than presenting a system of logic that undercuts critical concerns about the epistemic innocence of the proof-theory needed for Frege's Theorem, Heck has isolated, rather, the most basic, purest form of the tension, freed from the noise associated with the much greater expressive resources and limitless levels of impredicativity involved in the logic of *Grundgesetze*. Until that case is rebutted, the suggestion survives that Arché logic, no less than Frege's logic, is tainted by its partnership in paradox.

§6 *Does Arché Logic do better on Impredicativity?*

We can set that to one side, though. For I think it is clear that Arché logic with the unrestricted substitution rule must inherit the concerns about determinacy of sense and impredicativity that Heck themself presses against nominalist-neutralism and Hale's definable property proposal. Instances of the A-schemata are supposed to fix the meaning, that is, the satisfaction-conditions of the predicates for whose introduction and elimination they provide. There is no reason to doubt that they can do so when the range of *F* in their respective premises and conclusions is restricted to predicates whose satisfaction-conditions are *independently* determinate. But, as the reader will likely be tired of being reminded, such a restriction will block the derivation of Frege's Theorem — and specifically the derivation of arithmetical induction via the schematic rules for the ancestral, where we need to be able to introduce predicates themselves containing the ancestral operator for the occurrences of '*F*' in the schematic rules.

Heck is of course vividly aware of this. Their response[36] is to insist that, if we understand schematic generality as *full* schematic generality, then the rules must apply to *any* predicate in good standing, including the very predicate they define. There can be no restrictions. And that is surely quite right — *provided that* a candidate predicate is indeed in good standing, that is, has a determinate sense in the first place. But the concern we have raised about impredicativity is a concern about exactly that. Suppose you are familiar with 0 and predecession and then are told that, for any object y, 0 stands in the ancestral of the predecession relation to y just in case y satisfies every predicate that 0 satisfies and which is transmitted across predecession. That will doubtless seem to make decent enough sense —up until the point when it is explained that, in order for this relationship to obtain, *that very predicate* must be included in the scope of "every predicate" in its own formulation, so that in order for 0 to ancestrally precede y, it is not enough if y satisfies every *other* predicate which 0 has and which is transmitted across predecession; no, folks, in addition it is required that y satisfies *this one* too. At that point, you may feel, — and arguably *ought* to feel —you simply haven't been told what the satisfaction condition of "P^*0y" *is.*

It is my impression that recent years may have seen a kind of "impredicativity fatigue" setting in among many philosophers of mathematics and mathematicians interested in foundations— an attitude of "Impredicativity again? Really? Goodness, we have all made up our minds about that. Unlikely there is anything new to say." — and a consequent willingness to park the epistemological issues I have been trying to explain on a philosophical siding. Many moreover are comfortable to defer to the apparent authority of the Gödelian claim that there simply is no well-conceived barrier to the good standing of the procedure of specifying an entity of a certain kind by means of a formulation that

[36] Heck (2020), p.45 and following.

quantifies over a range of entities that purportedly includes it. Whether such a procedure is in good order turns, Gödel suggested, on whether the relevant range of entities are thought of as pre-existing and 'out there' — cf. Frank Ramsey's example, "the tallest man in the room" —or whether one conceives the specification in question in constructive terms, in which case there is indeed a difficulty.[37]

Gödel's dismissal provides succour for the thought that if we are inclined to realism about the classical notion of power set, and to thinking of the semantic values of predicates as sets of objects, or n-tuples of objects, as so conceived, then there should be no objection to full classical impredicative comprehension, a fortiori none to the modest form impredicative definition licensed by Heck's A-schemata under unrestricted substitution. I wish to stress that although the Gödel /Ramsey thought strikes me as solid in general, it is quite unsuited to address the specific worry about higher-order predication I have here tried to develop. To the contrary, that worry in turn should trouble any complacency about the determinacy of the classical notion of power set. In order for an impredicatively quantified first-level predicate to pick out a determinate subset of an objectual domain, it has first to express a *determinate condition* by satisfaction of which membership of the relevant subset is determined. If there is a standing doubt about whether that is so, it cannot be addressed simply by invoking the classical realist notion of power set. The full classical range of subsets may indeed be 'out there' all right. But it cannot be assumed in general that predicates of the kind that are causing concern will pick out any particular one of them unless the worry that impredicative higher-order quantification spawns indeterminacy of satisfaction-conditions has *first* been addressed.

The crucial point is that even falling back on the classical power set of an infinite first-order domain to supply a second-order domain is of no avail in avoiding the central difficulty with impredicativity that I have been concerned with. This is because our basic problem, to repeat, is not with the very idea of second-order entities for which we may have no means of intelligible individuative specification, problematic though that idea may well be felt to be, but with the good-standing— determinate significance—precisely of what are supposed to *be* certain individuative specifications of such entities. It doesn't matter whether we take the semantic value of a first-level predicate to be a set, or a Fregean Concept, or a sack of potatoes, or whatever else, nor what we take to be the relationship of such entities to possibilities of intelligible predication. An impredicatively quantified first-level predicate, like any predicate, has a semantic value at all only if it expresses a well-defined satisfaction condition on the elements of the objectual domain, and our concern has been— if I may be forgiven for harping on it —whether, when *actively* impredicative second-order quantification is essentially involved, any determinate such satisfaction condition is indeed expressed. If that worry is soundly conceived, it is an utterly irrelevant response to invoke the idea that the elements of the second-order domain may be 'out there' yet transcend finitely intelligible expression. The worry has to do with whether predicates formed by actively impredicative quantification possess a determinate sense; it is only *after* that has been addressed that questions about the kind of reference they may have, whether the kinds of entity concerned may transcend possible predication, etc., can so much as arise.

§7 *Generic quantification?*

The problem I have been urging arises if we think of linguistic meaning at large in *molecular* terms; that is, if (i) we think of the meaning of compound expressions generally as functionally dependent on the independently determined meanings of their constituents

[37] Gödel (1944)

and (ii) we think of the instances of quantified statements, while not literally constituents of them, as standing to them in an analogous relation of semantic priority, so that the determination of the truth-conditions of quantified statements is a function of those of their constituents. A defence of the kind of use made of impredicative higher-order quantification required by neo-logicism must reject this molecularism. It must develop an opposed kind of semantic *holism*, whereby the image of meaning invariably flowing upwards, as it were, from simpler to more complex admits of principled exceptions, so that predicates formed by actively impredicative higher-order quantifications may after all possess a fully determinate sense, licensing the distinctive inferential moves that the neo-logicist requires.

It is notable that such a 'top-down' or 'generic' conception of, at least, universal quantification is prefigured in recent writings by a number of philosophers.[38] The basic idea is that the truth a universally quantified statement may — and in certain cases, should — be viewed as grounded *intensionally*: that is, as sourced in some shared essential character of the elements of the domain over which the quantification is made, irrespective of the actual constitution of that domain, or, in case of restricted quantification — All F's are G — as grounded in an internal relation between being F and being G, irrespective of the actual extension of F. This idea is, notably, available to construe the sense of generalisations over domains — the ordinals, for example — whose extensions may be viewed as, in some sense or other, indeterminate. It is arguably a key component of one of Michael Dummett's arguments for an intuitionistic understanding of quantification over 'indefinitely extensible' mathematical domains. Is it of any assistance in the present context?

I am going to end on a very cautiously optimistic note. Bearing in mind that the generic conception has no evident application beyond the case of universal quantification, consider any impredicative higher-order predicate of objects, "$\forall X(\Phi X \longrightarrow Xy)$", expressing the condition of an object's having every Φ-property. What is the satisfaction condition of such a predicate when its ingredient quantifier is construed generically? Roughly this: that y is to be such that it satisfies any condition which is Φ just in virtue of that condition's being Φ and no matter which are the conditions which actually are Φ. What is wanted, then, in order that y should have such a property, is that there be some essential relationship between y and having properties that are Φ. And if there is indeed such a relationship, then it ought to ground y's having that very property if it too is Φ: there is now no obvious threat of circularity or regress in the explication of the satisfaction conditions of the impredicatively quantified predicate.

As remarked, the generic conception, as so far explicated, applies only to universal quantification. However, the neo-logicist construction of Peano arithmetic needs recourse only to Π_1^1 comprehension. So, as far as arithmetic at least is concerned, our question now is: is such a generic conception of higher-order universal quantification able to license all the higher-order impredicative definitions and moves involved in the normal proofs of Frege's Theorem? Is this broad conception of the satisfaction-conditions of the higher-order impredicatively quantified predicates that feature in the definitions and inferential steps that are involved, adequate for the jusitification of those definitions and inferential steps?

I believe the question is quite open. Certainly, to the best of my knowledge, none of those who have so far contributed, either as critics or as friends, to the neo-logicist programme has yet provided a clear reasoned answer, positive or negative. If one is

[38] See for instance Hale (2020), Wright (2018) and Linnebo (2018) to whom the terminology of 'generic' vs. 'instance-based' conceptions of quantification is due.

possible, then, as Hume said, we " . . . can only desire, that [the] reasoning may be produc'd, in order to be expos'd to our examination".[39]

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