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Multiscale matrix pencils for separable reconstruction problems

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Abstract

The nonlinear inverse problem of exponential data fitting is separable since the fitting function is a linear combination of parameterized exponential functions, thus allowing to solve for the linear coefficients separately from the nonlinear parameters. The matrix pencil method, which reformulates the problem statement into a generalized eigenvalue problem for the nonlinear parameters and a structured linear system for the linear parameters, is generally considered as the more stable method to solve the problem computationally. In Section 2 the matrix pencil associated with the classical complex exponential fitting or sparse interpolation problem is summarized and the concepts of dilation and translation are introduced to obtain matrix pencils at different scales. Exponential analysis was earlier generalized to the use of several polynomial basis functions and some operator eigenfunctions. However, in most generalizations a computational scheme in terms of an eigenvalue problem is lacking. In the subsequent Sections 3–6 the matrix pencil formulation, including the dilation and translation paradigm, is generalized to more functions. Each of these periodic, polynomial or special function classes needs a tailored approach, where optimal use is made of the properties of the parameterized elementary or special function used in the sparse interpolation problem under consideration. With each generalization a structured linear matrix pencil is associated, immediately leading to a computational scheme for the nonlinear and linear parameters, respectively from a generalized eigenvalue problem and one or more structured linear systems. Finally, in Section 7 we illustrate the new methods.

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1 Introduction

The nonlinear inverse problems of complex exponential analysis [1, 2] and sparse polynomial interpolation [3, 4] from uniformly sampled values can both be traced back to the exponential fitting method of de Prony from the eighteenth century [5, 6]:

$$f_j := f(t_j) = \sum_{i=1}^{n} \alpha_i \exp(\phi_i t_j), \alpha_i, \phi_i \in \mathbb{R}, \quad t_j = j\Delta \in \mathbb{R}, j = 0, \ldots, 2n-1. \tag{1}$$

The French nobleman de Prony solved the problem by obtaining the n nonlinear parameters ϕ_i from the roots of a polynomial and the n coefficients α_i as the solution of a Vandermonde structured linear system. Almost 200 years later this basic fitting problem, which plays an important role [7, 8] in many computational science disciplines, engineering applications and digital signal processing, was reformulated in terms of a generalized eigenvalue problem [9]. This reformulation, which is referred to as the matrix pencil method, is generally the most reliable one when solving the exponential analysis problem computationally.

It is the property

$$\exp(\phi_i t_{j+1}) = \exp(\phi_i \Delta) \exp(\phi_i t_j)$$

of the building blocks $\exp(\phi_i t)$ in (1) that allows to split the nonlinear interpolation problem (1) into two numerical linear algebra problems, namely the separate computation of the nonlinear parameters ϕ_i from a generalized eigenvalue problem on the one hand and the linear coefficients α_i from a structured linear system on the other.

Problem statement (1) was partially generalized, to the use of non-standard polynomial bases such as the Pochhammer basis and Chebyshev and Legendre polynomials [10–14] and to the use of some eigenfunctions of linear operators [15–17]. Many of these generalizations are unified in the algebraic framework described in [18].

What is lacking in most of the generalizations above, is a reformulation in terms of numerical linear algebra problems. In this paper we carry the generalized eigenvalue formulation of (1), so essentially the matrix pencil method, to linear combinations of the trigonometric functions cosine, sine, the hyperbolic cosine and sine functions, the Chebyshev (1st, 2nd, 3rd, 4th kind) and spread polynomials, the Gaussian function, the sinc and gamma function. In addition, we introduce the paradigm of a selectable dilation σ and translation τ of the interpolation points, as used in refinable function theory. All of the above functions namely satisfy a property similar to

$$\exp(\phi_i t_{\tau+(j+1)\sigma}) = \exp(\phi_i t_\tau) \exp^\sigma(\phi_i \Delta) \exp(\phi_i t_{j\sigma}),$$

which allows to separate the effect of the scale σ and the shift τ on the estimation of the parameters ϕ_i and α_i. This multiscale option will prove to be useful in several situations, as further detailed in Section 2.2.

In each of the subsequent sections on the trigonometric and hyperbolic functions, polynomial functions, the Gaussian distribution, and some special functions, a different approach is required to express the nonlinear inverse problem

$$f_j = \sum_{i=1}^{n} \alpha_i g(\phi_i; t_j), \qquad \alpha_i, \phi_i \in \mathbb{C}, \quad t_j \in \mathbb{R} \tag{2}$$

under consideration, as a generalized eigenvalue problem, tailored to the particular properties of the building block $g(\phi_i; t)$ in use. The interpolant is always computed directly from the evaluations f_j where the t_j follow some regular interpolation point pattern associated with the specific function $g(\phi_i; t)$.

2 Exponential fitting

We first lay out how the whole theory works for the exponential problem, where $g(\phi_i; t) = \exp(\phi_i t)$.

2.1 Scale and shift scheme

By a combination of [9] and [19] we obtain the following. Let $f(t)$ be given by

$$f(t) = \sum_{i=1}^{n} \alpha_i \exp(\phi_i t), \qquad \alpha_i, \phi_i \in \mathbb{C} \tag{3}$$

and let us sample $f(t)$ at the equidistant points $t_j = j\Delta$ for $j = 0, 1, 2, \ldots$ with $\Delta \in \mathbb{R}^+$, or more generally at $t_{\tau+j\sigma} = (\tau + j\sigma)\Delta$ with $\sigma \in \mathbb{N}$ and $\tau \in \mathbb{Z}$, where the frequency content in (3) is limited by [20, 21]

$$|\Im(\phi_i)|\Delta < \pi, \qquad i = 1, \ldots, n, \tag{4}$$

with $\Im(\cdot)$ denoting the imaginary part. More generally, σ and τ can belong to $\mathbb{Q}^+$ and $\mathbb{Q}$ respectively, as discussed in Section 2.5. The values σ and τ are called the scaling factor and shift term respectively. We denote the collected samples by

$$f_{\tau+j\sigma} := f(t_{\tau+j\sigma}), \qquad j = 0, 1, 2, \ldots$$

From $\exp(\phi_i t_{j+1}) = \exp(\phi_i \Delta) \exp(\phi_i t_j)$ we find that

$$f_{j+1} = \sum_{i=1}^{n} \alpha_i \exp(\phi_i \Delta) \exp(\phi_i j\Delta),$$

or more generally for $\sigma \in \mathbb{N}$ and $\tau \in \mathbb{Z}$ that

$$f_{\tau+j\sigma} = \sum_{i=1}^{n} \alpha_i \exp(\phi_i \tau \Delta) \exp(\phi_i j\sigma \Delta). \tag{5}$$

Hence we see that the scaling σ and the shift τ are separated in a natural way when evaluating (3) at $t_{\tau+j\sigma}$, a property that plays an important role in the sequel. The freedom to choose σ and τ when setting up the sampling scheme, allows to stretch, shrink and translate the otherwise uniform progression of sampling points dictated by the sampling step Δ.

The aim is now to estimate the model order n, and the parameters $\phi_1, \ldots, \phi_n$ and $\alpha_1, \ldots, \alpha_n$ in (3) from samples f_j at a selection of points t_j.

2.2 Generalized eigenvalue formulation

In this and the next subsection we assume for a moment that n was determined. With $n, \sigma \in \mathbb{N}$, $\tau \in \mathbb{Z}$ we define

$${}_{\sigma}^{\tau}H_n := \begin{pmatrix} f_\tau & f_{\tau+\sigma} & \cdots & f_{\tau+(n-1)\sigma} \\ f_{\tau+\sigma} & & & \\ \vdots & & \iddots & \vdots \\ f_{\tau+(n-1)\sigma} & & \cdots & f_{\tau+(2n-2)\sigma} \end{pmatrix}. \tag{6}$$

It is well-known that the Hankel matrix ${}_{\sigma}^{\tau}H_n$ can be decomposed as

$$\begin{aligned} {}_{\sigma}^{\tau}H_n &= V_n \Lambda_n A_n V_n^T, \\ V_n &= \begin{pmatrix} 1 & \cdots & 1 \\ \exp(\phi_1 \sigma \Delta) & \cdots & \exp(\phi_n \sigma \Delta) \\ \vdots & & \vdots \\ \exp(\phi_1 (n-1)\sigma\Delta) & \cdots & \exp(\phi_n (n-1)\sigma\Delta) \end{pmatrix}, \\ A_n &= \operatorname{diag}(\alpha_1, \ldots, \alpha_n), \\ \Lambda_n &= \operatorname{diag}(\exp(\phi_1 \tau \Delta), \ldots, \exp(\phi_n \tau \Delta)). \end{aligned} \tag{7}$$

This decomposition on the one hand translates (5) and on the other hand connects it to a generalized eigenvalue problem: the values $\exp(\phi_i \sigma \Delta)$ can be retrieved [9] as the generalized eigenvalues of the problem

$$\left({}_{\sigma}^{\sigma}H_n\right) v_i = \exp(\phi_i \sigma \Delta) \left({}_{\sigma}^{0}H_n\right) v_i, \qquad i = 1, \ldots, n, \tag{8}$$

where v_i are the generalized right eigenvectors. Setting up this generalized eigenvalue problem requires the $2n$ samples $f_{j\sigma}$, $j = 0, \ldots, 2n-1$. A similar statement holds for the values $\exp(\phi_i \tau \Delta)$ from the linear pencil $({}_{\sigma}^{\tau}H_n, {}_{\sigma}^{0}H_n)$. In [5, 9] the choices $\sigma = 1$

and $\tau = 1$ are made and then, from the generalized eigenvalues $\exp(\phi_i \Delta)$, the complex numbers ϕ_i can be retrieved uniquely because of the restriction $|\Im(\phi_i)|\Delta < \pi$.

Choosing $\sigma > 1$ offers a number of advantages though, among which:

- reconditioning [19, 22, 23] of a possibly ill-conditioned problem statement,
- superresolution [19, 24] in the case of clustered frequencies,
- validation [25] of the exponential analysis output for n and ϕ_i, $i = 1, \ldots, n$,
- the possibility to parallelize the estimation of the parameters ϕ_i [25].

With $\sigma > 1$ the ϕ_i cannot necessarily be retrieved uniquely from the generalized eigenvalues $\exp(\phi_i \sigma \Delta)$ since $|\Im(\phi_i)| \sigma \Delta$ may well be larger than π. Let us indicate how to solve that problem which is called aliasing.

2.3 Vandermonde structured linear systems

For chosen σ and with $\tau = 0$, the α_i are computed from the interpolation conditions

$$\sum_{i=1}^{n} \alpha_i \exp(\phi_i t_{j\sigma}) = f_{j\sigma}, \qquad j = 0, \ldots, 2n-1, \qquad \sigma \in \mathbb{N}, \tag{9}$$

either by solving the system in the least squares sense, in the presence of noise, or by solving a subset of n interpolation conditions in the case of noiseless samples. The samples of $f(t)$ occurring in (9) are the same samples as the ones used to fill the Hankel matrices in (8) with. Note that

$$\exp(\phi_i t_{j\sigma}) = (\exp(\phi_i \sigma \Delta))^j \,,$$

and that for fixed σ the coefficient matrix of (9) is therefore a transposed Vandermonde matrix with nodes $\exp(\phi_i \sigma \Delta)$. In a noisy context the Hankel matrices in (8) can also be extended to rectangular $N \times \nu$ matrices with $N > \nu \geq n$ and the generalized eigenvalue problem can be considered in a least squares sense [26]. In that case the index j in (9) runs from 0 to $N + \nu - 1$.

Next, for chosen nonzero τ, a shifted set of at least n samples $f_{\tau + j\sigma}$ is interpreted as

$$f_{\tau+j\sigma} = \sum_{i=1}^{n} (\alpha_i \exp(\phi_i \tau \Delta)) \exp(\phi_i j \sigma \Delta), \qquad j = k, \ldots, k+n-1, \qquad \tau \in \mathbb{Z}, \tag{10}$$

where $k \in \{0, 1, \ldots, n\}$ is fixed. Note that (10) is merely a shifted version of the original problem (3), where the effect of the shift is pushed into the coefficients of (3). The latter is possible because of (5). From (10), having the same (but maybe less) coefficient matrix entries as (9), we compute the unknown coefficients $\alpha_i \exp(\phi_i \tau \Delta)$. From α_i and $\alpha_i \exp(\phi_i \tau \Delta)$ we then obtain

$$\frac{\alpha_i \exp(\phi_i \tau \Delta)}{\alpha_i} = \exp(\phi_i \tau \Delta),$$

from which again the ϕ_i cannot necessarily be extracted unambiguously if $\tau > 1$. But the following can be proved [19].

Denote $s_{i,\sigma} := \text{sign}\left(\Im\left(\text{Ln}\left(\exp(\phi_i \sigma \Delta)\right)\right)\right)$ and $s_{i,\tau} := \text{sign}\left(\Im\left(\text{Ln}\left(\exp(\phi_i \tau \Delta)\right)\right)\right)$, where $\text{Ln}(\cdot)$ indicates the principal branch of the complex natural logarithm and $|\Im\left(\text{Ln}\left(\exp(\phi_i \sigma \Delta)\right)\right)| \leq \pi$. If $\gcd(\sigma, \tau) = 1$, then the sets

$$S_i = \left\{ \frac{1}{\sigma\Delta} \text{Ln}\left(\exp(\phi_i \sigma \Delta)\right) + \frac{2\pi \mathtt{i}}{\sigma\Delta}\ell,\ \ell = -s_{i,\sigma} \lfloor \sigma/2 \rfloor, \ldots, 0, \ldots, s_{i,\sigma}(\lceil \sigma/2 \rceil - 1) \right\},$$

$$T_i = \left\{ \frac{1}{\tau\Delta} \text{Ln}\left(\exp(\phi_i \tau \Delta)\right) + \frac{2\pi \mathtt{i}}{\tau\Delta}\ell,\ \ell = -s_{i,\tau} \lfloor \tau/2 \rfloor, \ldots, 0, \ldots, s_{i,\tau}(\lceil \tau/2 \rceil - 1) \right\},$$

which contain all the possible arguments for ϕ_i in $\exp(\phi_i \sigma \Delta)$ from (8) and in $\exp(\phi_i \tau \Delta)$ from (10) respectively, have a unique intersection [19]. How to obtain this unique element in the intersection and identify the ϕ_i is detailed in [19, 25]. Convenient choices for σ and τ depend somewhat on the noise level and their selection is also discussed in [25].

So at this point the nonlinear parameters $\phi_i, i = 1, \ldots, n$ and the linear $\alpha_i, i = 1, \ldots, n$ in (3) are computed through the solution of (8) and (9), and if $\sigma > 1$ also (10). Remains to discuss how to determine n.

2.4 Determining the sparsity

What can be said about the number of terms n in (3), which is also called the sparsity? From [27, p. 603] and [28] we know for general σ that

$$\begin{aligned} &\det {}_{\sigma}^{0}H_\nu = 0 \text{ only accidentally}, \qquad \nu < n, \\ &\det {}_{\sigma}^{0}H_n \neq 0, \\ &\det {}_{\sigma}^{0}H_\nu = 0, \qquad \nu > n. \end{aligned} \tag{11}$$

The regularity of ${}_{\sigma}^{0}H_n$ persists for any value of Δ when collecting the samples to fill the matrix with, while an accidental singularity of ${}_{\sigma}^{0}H_\nu$ with $\nu < n$ only occurs for an unfortunate choice of Δ that makes the determinant zero. A standard approach to make use of this statement is to compute a singular value decomposition of the Hankel matrix ${}_{\sigma}^{0}H_\nu$ and this for increasing values of ν. In the presence of noise and/or clustered eigenvalues, this technique is not always reliable and we need to consider rather large values of ν for a correct estimate of n [24] or turn our attention to some validation add-on [25].

With $\sigma = 1$ and in the absence of noise, the exponential fitting problem can be solved from $2n$ samples for $\alpha_1, \ldots, \alpha_n$ and $\phi_1, \ldots, \phi_n$ and at least one additional sample to confirm n. As pointed out already, it may be worthwhile to take $\sigma > 1$ and throw in at least an additional n values $f_{\tau + j\sigma}$ to remedy the aliasing. Moreover, if $\max_{i=1,\ldots,n} |\Im(\phi_i)|$ is quite large, then Δ may become so small that collecting the samples f_j becomes expensive and so it may be more feasible to work with a larger sampling interval $\sigma\Delta$.

2.5 Computational variants

Besides having $\sigma \in \mathbb{N}$ and $\tau \in \mathbb{Z}$, more general choices are possible. An easy practical generalization is when the scale factor and shift term are rational numbers $\sigma/\rho_1 \in \mathbb{Q}^+$ and $\tau/\rho_2 \in \mathbb{Q}$ respectively, with $\sigma, \rho_1, \rho_2 \in \mathbb{N}$ and $\tau \in \mathbb{Z}$. In that case the condition $\gcd(\sigma, \tau) = 1$ for S_i and T_i to have a unique intersection, is replaced by $\gcd(\overline{\sigma}, \overline{\tau}) = 1$ where $\sigma/\rho_1 = \overline{\sigma}/\rho, \tau/\rho_2 = \overline{\tau}/\rho$ with $\rho = \text{lcm}(\rho_1, \rho_2)$.

We remark that, although the sparse interpolation problem can be solved from the $2n$ samples $f_j, j = 0, \ldots, 2n-1$ when $\sigma = 1$, we need at least an additional n samples at the shifted locations $t_{\tau+j\sigma}, j = k, \ldots, k+n-1$ when $\sigma > 1$. The former is Prony's original problem statement in [5] and the latter is the generalization presented in [19]. The factorization (7) allows some alternative computational schemes, which may deliver a better numerical accuracy but demand somewhat more samples.

First we remark that the use of a shift τ can of course be replaced by the choice of a second scale factor $\tilde{\sigma}$ relatively prime with σ. But this option requires the solution of two generalized eigenvalue problems of which the generalized eigenvalues need to be matched in a combinatorial step. Also, the sampling scheme looks different and requires the $4n-1$ sampling points

$$\{t_{j\sigma}, 0 \le j \le 2n-1\} \cup \{t_{j\tilde{\sigma}}, 0 \le j \le 2n-1\}, \qquad \gcd(\sigma, \tilde{\sigma}) = 1.$$

A better option is to set up the generalized eigenvalue problem

$$ {}_{\sigma}^{\tau} H_n v_i = \exp(\phi_i \tau \Delta) {}_{\sigma}^{0} H_n v_i, \qquad i = 1, \ldots, n \tag{12}$$

which in a natural way connects each eigenvalue $\exp(\phi_i \tau \Delta)$, bringing forth the set T_i, to its associated eigenvector v_i bringing forth the set S_i. The latter is derived from the quotient of any two consecutive entries in the vector ${}_{\sigma}^{0} H_n v_i$ which is a scalar multiple of

$$\alpha_i (1, \exp(\phi_i \sigma \Delta), \ldots, \exp(\phi_i (n-1)\sigma \Delta))^T.$$

Such a scheme requires the $4n-2$ samples

$$\{t_{j\sigma}, 0 \le j \le 2n-2\} \cup \{t_{\tau+j\sigma}, 0 \le j \le 2n-2\}, \qquad \gcd(\sigma, \tau) = 1.$$

Note that the generalized eigenvectors v_i are actually insensitive to the shift τ: the eigenvectors of (8) and (12) are identical. This is a remarkable fact that reappears in each of the subsequent (sub)sections dealing with other choices for $g(\phi_i; t)$.

We now turn our attention to the identification of other families of parameterized functions and patterns of sampling points. We distinguish between trigonometric and hyperbolic, polynomial and other important functions. Our focus here is on the derivation of the mathematical theory and not on the practical aspects of the numerical computation.

3 Trigonometric functions

The generalized eigenvalue formulation (8) incorporating the scaling parameter σ, was generalized to $g(\phi_i; t) = \cos(\phi_i t)$ in [11] for integer ϕ_i only. Here we present a more elegant full generalization for $\cos(\phi_i t)$ including the use of a shift τ as in (10) to restore uniqueness of the solution if necessary. In addition we generalize the scale and shift approach to the functions sine, cosine hyperbolic and sine hyperbolic.

3.1 Cosine function

Let $g(\phi_i; t) = \cos(\phi_i t)$ with $\phi_i \in \mathbb{R}$ where

$$|\phi_i|\Delta < \pi, \qquad i = 1, \dots, n. \tag{13}$$

Since $\cos(\phi_i t) = \cos(-\phi_i t)$, we are only interested in the $|\phi_i|$, $i = 1, \dots, n$, disregarding the sign of each ϕ_i. With $t_j = j\Delta$ we still denote

$$f_{\tau+j\sigma} := \sum_{i=1}^{n} \alpha_i \cos(\phi_i(\tau + j\sigma)\Delta), \tag{14}$$

and because of

$$\frac{1}{2}\cos(\phi_i t_{j+1}) + \frac{1}{2}\cos(\phi_i t_{j-1}) = \cos(\phi_i \Delta)\cos(\phi_i t_j) \tag{15}$$

we now also introduce for fixed chosen σ and τ,

$$F_{\tau+j\sigma} := F(\sigma, \tau; t_j) = \frac{1}{2} f_{\tau+j\sigma} + \frac{1}{2} f_{\tau-j\sigma}, = \sum_{i=1}^{n} \alpha_i \cos(\phi_i \tau \Delta)\cos(\phi_i j\sigma\Delta). \tag{16}$$

Relation (15) deals with the case $\sigma = 1$ and $\tau = 1$, while the expression $F_{\tau+j\sigma}$ is a generalization of (15) for general σ and τ. Observe the achieved separation in (16) of the scaling σ and the shift τ. We emphasize that σ and τ are fixed before defining the $F(\sigma, \tau; t_j)$. Otherwise the index j cannot be associated uniquely with the value $1/2(f_{\tau+j\sigma} + f_{\tau-j\sigma})$.

Besides the Hankel structured ${}_{\sigma}^{\tau}H_n$, we introduce the Toeplitz structured

$${}_{\sigma}^{\tau}T_n := \begin{pmatrix} f_\tau & f_{\tau-\sigma} & \cdots & f_{\tau-(n-1)\sigma} \\ f_{\tau+\sigma} & & & \\ \vdots & & \ddots & \vdots \\ f_{\tau+(n-1)\sigma} & & \cdots & f_\tau \end{pmatrix},$$

which is symmetric when $\tau = 0$. Now consider the structured matrix

$${}_{\sigma}^{\tau}C_n := \frac{1}{4}\left({}_{\sigma}^{\tau}H_n\right) + \frac{1}{4}\left({}_{-\sigma}^{\tau}H_n\right) + \frac{1}{4}\left({}_{\sigma}^{\tau}T_n\right) + \frac{1}{4}\left({}_{-\sigma}^{\tau}T_n\right), \tag{17}$$

where ${}^{\tau}_{-\sigma}T_n = {}^{\tau}_{\sigma}T_n^T$. When $\tau = 0$, the first two matrices in the sum coincide and the latter two do as well. Note that working directly with the cosine function instead of expressing it in terms of the exponential as $\cos x = (\exp(\mathtt{i}x)+\exp(-\mathtt{i}x))/2$, reduces the size of the matrices involved in the pencil from $2n$ to n.

Theorem 1 *The matrix ${}^{\tau}_{\sigma}C_n$ factorizes as*

$$\begin{aligned}
{}^{\tau}_{\sigma}C_n &= W_n L_n A_n W_n^T, \\
W_n &= \begin{pmatrix} 1 & \cdots & 1 \\ \cos(\phi_1 \sigma \Delta) & \cdots & \cos(\phi_n \sigma \Delta) \\ \vdots & & \vdots \\ \cos(\phi_1 (n-1)\sigma \Delta) & \cdots & \cos(\phi_n (n-1)\sigma \Delta) \end{pmatrix}, \\
A_n &= \mathrm{diag}(\alpha_1, \ldots, \alpha_n), \\
L_n &= \mathrm{diag}(\cos(\phi_1 \tau \Delta), \ldots, \cos(\phi_n \tau \Delta)).
\end{aligned}$$

Proof The proof is a verification of the matrix product entry at position $(k+1, \ell+1)$ for $k, \ell = 0, \ldots, n-1$:

$$\begin{aligned}
&1/4 f_{\tau+(k+\ell)\sigma} + 1/4 f_{\tau-(k+\ell)\sigma} + 1/4 f_{\tau+(k-\ell)\sigma} + 1/4 f_{\tau+(-k+\ell)\sigma} \\
&= 1/2 \sum_{i=1}^{n} \alpha_i \cos(\phi_i \tau \Delta) \cos(\phi_i (k+\ell)\sigma \Delta) + 1/2 \sum_{i=1}^{n} \alpha_i \cos(\phi_i \tau \Delta) \cos(\phi_i (k-\ell)\sigma \Delta) \\
&= \sum_{i=1}^{n} \alpha_i \cos(\phi_i \tau \Delta) \cos(\phi_i k \sigma \Delta) \cos(\phi_i \ell \sigma \Delta).
\end{aligned}$$

□

This matrix factorization translates (16) and opens the door to the use of a generalized eigenvalue problem: the cosine equivalent of (8) becomes

$$\left({}^{\sigma}_{\sigma}C_n\right) v_i = \cos(\phi_i \sigma \Delta) \left({}^{0}_{\sigma}C_n\right) v_i, \qquad i = 1, \ldots, n, \tag{18}$$

where v_i are the generalized right eigenvectors. Setting up (18) takes $2n$ evaluations $f_{j\sigma}$, as in the exponential case. Before turning our attention to the extraction of the ϕ_i from the generalized eigenvalues $\cos(\phi_i \sigma \Delta)$, we solve two structured linear systems of interpolation conditions.

The coefficients α_i in (14) are computed from

$$\sum_{i=1}^{n} \alpha_i \cos(\phi_i j \sigma \Delta) = f_{j\sigma}, \qquad j = 0, \ldots, 2n-1, \quad \sigma \in \mathbb{N}.$$

Making use of (16), the coefficients $\alpha_i \cos(\phi_i \tau \Delta)$ are obtained from the shifted interpolation conditions

$$\sum_{i=1}^{n} (\alpha_i \cos(\phi_i \tau \Delta)) \cos(\phi_i j\sigma \Delta) = F_{\tau+j\sigma}, \qquad j = k, \ldots, k+n-1, \quad \tau \in \mathbb{Z}, \quad (19)$$

where $k \in \{0, 1, \ldots, n\}$ is fixed. While for $\sigma = 1$ the sparse interpolation problem can be solved from $2n$ samples taken at the points $t_j = j\Delta$, $j = 0, \ldots, 2n-1$, for $\sigma > 1$ additional samples are required at the shifted locations $t_{\tau \pm j\sigma} = (\tau \pm j\sigma)\Delta$ in order to resolve the ambiguity that arises when extracting the nonlinear parameters ϕ_i from the values $\cos(\phi_i \sigma \Delta)$. The quotient

$$\frac{\alpha_i \cos(\phi_i \tau \Delta)}{\alpha_i}, \qquad i = 1, \ldots, n$$

delivers the values $\cos(\phi_i \tau \Delta), i = 1, \ldots, n$. Neither from $\cos(\phi_i \sigma \Delta)$ nor from $\cos(\phi_i \tau \Delta)$ the parameters ϕ_i can necessarily be extracted uniquely when $\sigma > 1$ and $\tau > 1$. But the following result is proved in the Appendix.

If $\gcd(\sigma, \tau) = 1$, the sets

$$S_i = \left\{ \frac{1}{\sigma \Delta} \operatorname{Arccos}\left(\cos(\phi_i \sigma \Delta)\right) + \frac{2\pi}{\sigma \Delta}\ell, \ \ell = -\lfloor \sigma/2 \rfloor, \ldots, 0, \ldots, \lceil \sigma/2 \rceil - 1 \right\},$$

$$T_i = \left\{ \frac{1}{\tau \Delta} \operatorname{Arccos}\left(\cos(\phi_i \tau \Delta)\right) + \frac{2\pi}{\tau \Delta}\ell, \ \ell = -\lfloor \tau/2 \rfloor, \ldots, 0, \ldots, \lceil \tau/2 \rceil - 1 \right\}$$

containing all the candidate arguments for ϕ_i in $\cos(\phi_i \sigma \Delta)$ and $\cos(\phi_i \tau \Delta)$ respectively, have at most two elements in their intersection. Here $\operatorname{Arccos}(\cdot) \in [0, \pi]$ denotes the principal branch of the arccosine function. In case two elements are found, then it suffices to extend (19) to

$$\sum_{i=1}^{n} (\alpha_i \cos(\phi_i (\sigma + \tau)\Delta)) \cos(\phi_i j\sigma \Delta) = F_{(\sigma+\tau)+j\sigma}, \qquad j = k, \ldots, k+n-1,$$

which only requires the additional sample $f_{\tau+(k+n)\sigma}$ as $f_{\tau-(k+n-2)\sigma}$ is already available. From this extension, $\cos(\phi_i(\sigma + \tau)\Delta)$ can be obtained in the same way as $\cos(\phi_i \tau \Delta)$. As explained in the Appendix, only one of the two elements in the intersection of S_i and T_i fits the computed $\cos(\phi_i(\sigma + \tau)\Delta)$ since $\gcd(\sigma, \tau) = 1$ implies that also $\gcd(\sigma, \sigma + \tau) = 1 = \gcd(\tau, \sigma + \tau)$.

So the unique identification of the ϕ_i can require $2n - 1$ additional samples at the shifted locations $(\tau \pm j\sigma)\Delta$, $j = 0, \ldots n-1$ if the intersections $S_i \cap T_i$ are all singletons, or $2n$ additional samples, namely at $(\tau \pm j\sigma)\Delta$, $j = 0, \ldots, n-1$ and $(\tau + n\sigma)\Delta$ if at least one of the intersections $S_i \cap T_i$ is not a singleton.

The factorization in Theorem 1 immediately allows to formulate the following cosine analogue of (11).

Corollary 1 *For the matrix* ${}^{0}_{\sigma}C_n$ *defined in* (17) *holds that*

$$\mathit{rank}\,{}^{0}_{\sigma}C_\nu = n, \qquad \nu \geq n.$$

To round up our discussion, we mention that from the factorization in Theorem 1, it is clear that for the generalized eigenvector v_i from the different generalized eigenvalue problem

$$\left({}^{\tau}_{\sigma}C_n\right) v_i = \cos(\phi_i \tau \Delta) \left({}^{0}_{\sigma}C_n\right) v_i,$$

holds that ${}^{0}_{\sigma}C_n v_i$ is a scalar multiple of

$$\alpha_i \left(1, \cos(\phi_i \sigma \Delta), \ldots, \cos(\phi_i (n-1)\sigma \Delta)\right)^T.$$

This immediately leads to a computational variant of the proposed scheme, similar to the one given in Section 2.4 for the exponential function, requiring somewhat more samples though. Let us now turn our attention to other trigonometric functions.

3.2 Sine function

Let $g(\phi_i; t) = \sin(\phi_i t)$ and let (13) hold. With $t_j = j\Delta$ We denote

$$f_{\tau+j\sigma} := \sum_{i=1}^{n} \alpha_i \sin(\phi_i(\tau + j\sigma)\Delta),$$

and because of

$$\frac{1}{2}\sin(\phi_i t_{j+1}) + \frac{1}{2}\sin(\phi_i t_{j-1}) = \cos(\phi_i \Delta)\sin(\phi_i t_j), \qquad \Delta = t_{j+1} - t_j, \tag{20}$$

we introduce for fixed chosen σ and τ,

$$F_{\tau+j\sigma} := F(\sigma, \tau; t_j) = \frac{1}{2} f_{\tau+j\sigma} + \frac{1}{2} f_{-\tau+j\sigma} = \sum_{i=1}^{n} (\alpha_i \cos(\phi_i \tau \Delta)) \sin(\phi_i j \sigma \Delta). \tag{21}$$

We fill the matrices ${}^{\tau}_{\sigma}H_n$ and the Toeplitz matrices ${}^{\tau}_{\sigma}T_n$ and define

$${}^{\tau}_{\sigma}B_n := \frac{1}{4}\left({}^{\sigma+\tau}_{\sigma}H_n\right) + \frac{1}{4}\left({}^{\sigma-\tau}_{\sigma}H_n\right) + \frac{1}{4}\left({}^{\sigma+\tau}_{\sigma}T_n\right) + \frac{1}{4}\left({}^{\sigma-\tau}_{\sigma}T_n\right). \tag{22}$$

Theorem 2 *The structured matrix* ${}^{\tau}_{\sigma}B_n$ *factorizes as*

$${}^{\tau}_{\sigma}B_n = U_n L_n A_n W_n^T,$$

$$U_n = \begin{pmatrix} \sin(\phi_1\sigma\Delta) & \cdots & \sin(\phi_n\sigma\Delta) \\ \vdots & & \vdots \\ \sin(\phi_1 n\sigma\Delta) & \cdots & \sin(\phi_n n\sigma\Delta) \end{pmatrix}$$

$$W_n = \begin{pmatrix} 1 & \cdots & 1 \\ \cos(\phi_1\sigma\Delta) & \cdots & \cos(\phi_n\sigma\Delta) \\ \vdots & & \vdots \\ \cos(\phi_1(n-1)\sigma\Delta) & \cdots & \cos(\phi_n(n-1)\sigma\Delta) \end{pmatrix},$$

$$A_n = \mathrm{diag}(\alpha_1, \ldots, \alpha_n),$$

$$L_n = \mathrm{diag}(\cos(\phi_1\tau\Delta), \ldots, \cos(\phi_n\tau\Delta)).$$

Proof The proof is again a verification of the matrix product entry, at the position $(k, \ell+1)$ with $k = 1, \ldots, n$ and $\ell = 0, \ldots, n-1$:

$$\begin{aligned} &1/4 f_{\tau+(k+\ell)\sigma} + 1/4 f_{-\tau+(k+\ell)\sigma} + 1/4 f_{\tau+(k-\ell)\sigma} + 1/4 f_{-\tau+(k-\ell)\sigma} \\ &= 1/2 \sum_{i=1}^{n} \alpha_i \cos(\phi_i\tau\Delta)\sin(\phi_i(k+\ell)\sigma\Delta) + 1/2 \sum_{i=1}^{n} \alpha_i \cos(\phi_i\tau\Delta)\sin(\phi_i(k-\ell)\sigma\Delta) \\ &= \sum_{i=1}^{n} \alpha_i \cos(\phi_i\tau\Delta)\sin(\phi_i k\sigma\Delta)\cos(\phi_i\ell\sigma\Delta). \end{aligned}$$

□

Note that the factorization involves precisely the building blocks in the shifted evaluation (21) of the help function $F(\sigma, \tau; t)$. From this decomposition we find that the $\cos(\phi_i\sigma\Delta)$, $i = 1, \ldots, n$ are obtained as the generalized eigenvalues of the problem

$$\left({}^{\sigma}_{\sigma}B_n\right) v_i = \cos(\phi_i\sigma\Delta) \left({}^{0}_{\sigma}B_n\right) v_i, \qquad i = 1, \ldots, n.$$

We point out that setting up this generalized eigenvalue problem requires samples of $f(t)$ at the points $t_{(-n+1)\sigma}, \ldots, t_{2n\sigma}$. Since $f(t_{j\sigma}) = -f(t_{-j\sigma})$ and $f(0) = 0$ it costs $2n$ samples. Unfortunately, at this point we cannot compute the α_i, $i = 1, \ldots, n$ from the linear system of interpolation conditions

$$\sum_{i=1}^{n} \alpha_i \sin(\phi_i j\sigma\Delta) = f_{j\sigma}, \qquad j = 1, \ldots, 2n,$$

as we usually do, because we do not have the matrix entries $\sin(\phi_i j\sigma\Delta)$ at our disposal. It is however easy to obtain the values $\cos(\phi_i j\sigma\Delta)$ because $\cos(\phi_i j\sigma\Delta) = \cos(\pm j \,\mathrm{Arccos}(\cos(\phi_i\sigma\Delta)))$ where $\mathrm{Arccos}(\cos(\phi_i\sigma\Delta))$ returns the principal branch value. The proper way to proceed is the following.

From Theorem 2 we get ${}^{0}_{\sigma}B_n^T = W_n A_n U_n^T$. So we can obtain the $\alpha_i \sin(\phi_i \sigma \Delta)$ in the first column of $A_n U_n^T$ from the structured linear system

$$W_n \begin{pmatrix} \alpha_1 \sin(\phi_1 \sigma \Delta) \\ \vdots \\ \alpha_n \sin(\phi_n \sigma \Delta) \end{pmatrix} = \begin{pmatrix} b_{11} \\ \vdots \\ b_{1n} \end{pmatrix}, \tag{23}$$

where ${}^{0}_{\sigma}B_n = (b_{ij})_{i,j=1}^n$. From the generalized eigenvalues $\cos(\phi_i \sigma \Delta)$, $i = 1, \dots$ and the $\alpha_i \sin(\phi_i \sigma \Delta)$ we can now recursively compute for $j = 1, \dots, n$,

$$\alpha_i \sin(\phi_i j \sigma \Delta) = \alpha_i \sin(\phi_i (j-1) \sigma \Delta) \cos(\phi_i \sigma \Delta) + \cos(\phi_i (j-1) \sigma \Delta) \alpha_i \sin(\phi_i \sigma \Delta).$$

The system of shifted linear interpolation conditions

$$\sum_{i=1}^{n} (\alpha_i \cos(\phi_i \tau \Delta)) \sin(\phi_i j \sigma \Delta) = F_{\tau + j\sigma}, \qquad j = k, \dots, k+n-1, \quad 1 \le k \le n+1$$

can then be looked at as

$$\sum_{i=1}^{n} (\alpha_i \sin(\phi_i j \sigma \Delta)) \cos(\phi_i \tau \Delta) = F_{\tau + j\sigma}, \qquad j = k, \dots, k+n-1 \tag{24}$$

having a coefficient matrix with entries $\alpha_i \sin(\phi_i j \sigma \Delta)$ and unknowns $\cos(\phi_i \tau \Delta)$. In order to retrieve the ϕ_i uniquely from the values $\cos(\phi_i \sigma \Delta)$ and $\cos(\phi_i \tau \Delta)$ with $\gcd(\sigma, \tau) = 1$, one proceeds as in the cosine case. Finally, the α_i are obtained from the expressions $\alpha_i \sin(\phi_i \sigma \Delta)$ after plugging in the correct arguments ϕ_i in $\sin(\phi_i \sigma \Delta)$ and dividing by it. So compared to the previous sections, the intermediate computation of the α_i before knowing the ϕ_i, is replaced by the intermediate computation of the $\alpha_i \sin(\phi_i \sigma \Delta)$. In the end, the α_i are revealed in a division, without the need to solve an additional linear system.

From the factorization in Theorem 2, the following sine analogue of (11) follows immediately.

Corollary 2 *For the matrix* ${}^{0}_{\sigma}B_n$ *defined in* (22) *holds that*

$$\mathit{rank}\, {}^{0}_{\sigma}B_\nu = n, \qquad \nu \ge n.$$

For completeness we mention that one also finds from this factorization that for v_i in the generalized eigenvalue problem

$$\left({}^{\tau}_{\sigma}B_n\right) v_i = \cos(\phi_i \tau \Delta) \left({}^{0}_{\sigma}B_n\right) v_i,$$

holds that ${}^{0}_{\sigma}B_n v_i$ is a scalar multiple of

$$\alpha_i \left(\sin(\phi_i \sigma \Delta), \dots, \sin(\phi_i n \sigma \Delta)\right)^T.$$

3.3 Phase shifts in cosine and sine

It is possible to include phase shift parameters in the cosine and sine interpolation schemes. We explain how, by working out the sparse interpolation of

$$f(t) = \sum_{i=1}^{n} \alpha_i \sin(\phi_i t - \psi_i), \qquad \psi_i \in \mathbb{R}, \quad \alpha_i, \phi_i \in \mathbb{C}. \tag{25}$$

Since $\sin t = (\exp(\mathrm{i}t) - \exp(-\mathrm{i}t))\,/2\mathrm{i}$, we can write each term in (25) as

$$\alpha_i \sin(\phi_i t - \psi_i) = \frac{\alpha_i \exp(-\mathrm{i}\psi_i)}{2\mathrm{i}} \exp(\mathrm{i}\phi_i t) - \frac{\alpha_i \exp(\mathrm{i}\psi_i)}{2\mathrm{i}} \exp(-\mathrm{i}\phi_i t).$$

So the sparse interpolation of (25) can be solved by considering the exponential sparse interpolation problem

$$\sum_{i=1}^{2n} \beta_i \exp(\mathrm{i}\zeta_i t),$$

where $\beta_{2i-1} = \alpha_i \exp(-\mathrm{i}\psi_i)/(2\mathrm{i})$, $\beta_{2i} = -\alpha_i \exp(\mathrm{i}\psi_i)/(2\mathrm{i})$ and $\zeta_{2i-1} = \phi_i = -\zeta_{2i}$. The computation of the ϕ_i through the ζ_i remains separated from that of the α_i and ψ_i. The latter are obtained as

$$\begin{aligned} \tan \psi_i &= -\mathrm{i}\frac{\beta_{2i} + \beta_{2i-1}}{\beta_{2i} - \beta_{2i-1}}, \\ \alpha_i &= -(\beta_{2i} + \beta_{2i-1})/\sin(\psi_i) = -\mathrm{i}(\beta_{2i} - \beta_{2i-1})/\cos(\psi_i). \end{aligned}$$

3.4 Hyperbolic functions

For $g(\phi_i; t) = \cosh(\phi_i t)$ the computational scheme parallels that of the cosine and for $g(\phi_i; t) = \sinh(\phi_i t)$ that of the sine. We merely write down the main issues.

When $g(\phi_i; t) = \cosh(\phi_i t)$, let

$$f_{\tau+j\sigma} := \sum_{i=1}^{n} \alpha_i \cosh(\phi_i(\tau + j\sigma)\Delta)$$

and for fixed chosen σ and τ, let

$$F_{\tau+j\sigma} := \frac{1}{2} f_{\tau+j\sigma} + \frac{1}{2} f_{\tau-j\sigma} = \sum_{i=1}^{n} \alpha_i \cosh(\phi_i \tau \Delta) \cosh(\phi_i j\sigma\Delta).$$

Subsequently the definition of the structured matrix ${}_{\sigma}^{\tau}C_n$ is used and in the factorization of Theorem 1, the cosine function is everywhere replaced by the cosine hyperbolic function.

Similarly, when $g(\phi_i; t) = \sinh(\phi_i t)$, let

$$f_{\tau+j\sigma} := \sum_{i=1}^{n} \alpha_i \sinh(\phi_i(\tau + j\sigma)\Delta)$$

and for fixed chosen σ and τ, let

$$F_{\tau+j\sigma} := \frac{1}{2} f_{\tau+j\sigma} + \frac{1}{2} f_{-\tau+j\sigma} = \sum_{i=1}^{n} \alpha_i \cosh(\phi_i \tau \Delta) \sinh(\phi_i j\sigma \Delta).$$

Now the definition of the structured matrix ${}_{\sigma}^{\tau}B_n$ is used and in the factorization of Theorem 2 the occurrences of cos are replaced by cosh and those of sin by sinh.

4 Polynomial functions

The orthogonal Chebyshev polynomials were among the first polynomial basis functions to be explored for use in combination with a scaling factor σ, in the context of sparse interpolation in symbolic-numeric computing [11]. We elaborate the topic further for numerical purposes and for lacunary or supersparse interpolation, making use of the scale factor σ and the shift term τ. We also extend the approach to other polynomial bases and connect to generalized eigenvalue formulations.

4.1 Chebyshev 1st kind

Let $g(m_i; t) = T_{m_i}(t)$ of degree m_i, which is defined by

$$T_m(t) = \cos(m\theta), \qquad t = \cos(\theta), \quad -1 \le t \le 1,$$

and consider the interpolation problem

$$f(t_j) = \sum_{i=1}^{n} \alpha_i T_{m_i}(t_j), \qquad \alpha_i \in \mathbb{C}, \quad m_i \in \mathbb{N}. \tag{26}$$

The Chebyshev polynomials $T_m(t)$ satisfy the recurrence relation

$$T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), \qquad T_1(t) = t, \quad T_0(t) = 1$$

and the property

$$\frac{1}{2} T_{m_i}(t_{j+1}) + \frac{1}{2} T_{m_i}(t_{j-1}) = T_{m_i}(\cos\Delta) T_{m_i}(t_j).$$

With $0 \le m_1 < m_2 < \ldots < m_n < M$ we choose $t_j = \cos(j\Delta)$ where $0 < \Delta \le \pi/M$. Note that the points t_j are allowed to occupy much more general positions than in [11]. If M is extremely large and n is small, in other words if the polynomial is very sparse, then it is a good idea to recover the actual m_i, $i = 1, \ldots, n$ in two tiers as we explain now. Let $\gcd(\sigma, \tau) = 1$. We denote

$$f_{\tau+j\sigma} := \sum_{i=1}^{n} \alpha_i T_{m_i}(t_{\tau+j\sigma}) \tag{27}$$

and introduce for fixed σ and τ,

$$F_{\tau+j\sigma} := F(\sigma, \tau; t_j) = \frac{1}{2} f_{\tau+j\sigma} + \frac{1}{2} f_{\tau-j\sigma} = \sum_{i=1}^{n} \alpha_i T_{m_i}(\cos(\tau\Delta)) T_{m_i}(\cos(j\sigma\Delta))$$

in order to separate the effect of σ and τ in the evaluation. With the same matrices ${}_{\sigma}^{\tau}H_n$, ${}_{\sigma}^{\tau}T_n$ and ${}_{\sigma}^{\tau}C_n$ as in the cosine subsection, now filled with the $f_{\tau+j\sigma}$ from (27), the values $T_{m_i}(\cos(\sigma\Delta))$ are the generalized eigenvalues of the problem

$$({}_{\sigma}^{\sigma}C_n) v_i = T_{m_i}(\cos(\sigma\Delta)) ({}_{\sigma}^{0}C_n) v_i, \quad i = 1, \ldots, n. \tag{28}$$

From the values $T_{m_i}(\cos(\sigma\Delta)) = \cos(m_i\sigma\Delta)$ the integer m_i cannot necessarily be retrieved unambiguously. We need to find out which of the elements in the set

$$S_i = \left\{ \frac{\pm 1}{\sigma\Delta} \operatorname{Arccos}(\cos(m_i\sigma\Delta)) + \frac{2\pi}{\sigma\Delta}\ell,\ \ell = 0, \ldots, \sigma - 1 \right\} \cap \mathbb{Z}_M$$

is the one satisfying (26), where $\operatorname{Arccos}(\cos(m_i\sigma\Delta))/(\sigma\Delta) \leq M/\sigma$. Depending on the relationship between σ and M (relatively prime, generator, divisor, ...) the set S_i may contain one or more candidate integers for m_i evaluating to the same value $\cos(m_i\sigma\Delta)$. To resolve the ambiguity we consider the Vandermonde-like system for the α_i, $i = 1, \ldots, n$,

$$\sum_{i=1}^{n} \alpha_i T_{m_i}(\cos(j\sigma\Delta)) = f_{j\sigma}, \qquad j = 0, \ldots, 2n-1,$$

and the shifted problem

$$\sum_{i=1}^{n} \left(\alpha_i T_{m_i}(\cos(\tau\Delta))\right) T_{m_i}(\cos(j\sigma\Delta)) = F_{\tau+j\sigma}, \qquad j = k, \ldots, k+n-1, \quad \tau \in \mathbb{Z},$$

from which we compute the $\alpha_i T_{m_i}(\cos(\tau\Delta)) = \alpha_i \cos(m_i\tau\Delta)$. Then

$$\cos(m_i\tau\Delta) = T_{m_i}(\cos(\tau\Delta)) = \frac{\alpha_i T_{m_i}\cos(\tau\Delta)}{\alpha_i}, \qquad i = 1, \ldots, n.$$

If the intersection of the set S_i with the set

$$T_i = \left\{ \frac{\pm 1}{\tau\Delta} \operatorname{Arccos}(\cos(m_i\tau\Delta)) + \frac{2\pi}{\tau\Delta}\ell,\ \ell = 0, \ldots, \tau - 1 \right\} \cap \mathbb{Z}_M$$

is processed as in Section 3.1, then one can eventually identify the correct m_i. An illustration thereof is given in Section 7.3.

When replacing (28) by

$$({}_{\sigma}^{\tau}C_n)v_i = T_{m_i}(\cos(\tau\Delta))({}_{\sigma}^{0}C_n)v_i, \quad i = 1, \ldots, n,$$

we find that for v_i holds that ${}_{\sigma}^{0}C_n v_i$ is a scalar multiple of

$$\alpha_i \left(1, T_{m_i}(\cos\sigma\Delta), \ldots, T_{m_i}(\cos(n-1)\sigma\Delta)\right)^T.$$

This offers an alternative algorithm similar to the alternative in Section 3.1 on the cosine function.

4.2 Chebyshev 2nd, 3rd and 4th kind

While the Chebyshev polynomials $T_{m_i}(t)$ of the first kind are intrinsically related to the cosine function, the Chebyshev polynomials $U_{m_i}(t)$ of the second kind can be expressed using the sine function:

$$\begin{aligned} U_m(t) &= \frac{\sin((m+1)\theta)}{\sin\theta}, \qquad t = \cos\theta, \quad -1 < t < 1 \\ U_m(-1) &= (-1)^m(m+1), \qquad U_m(1) = m+1. \end{aligned}$$

Therefore the sparse interpolation problem

$$f(t_j) = \sum_{i=1}^{n} \alpha_i U_{m_i}(t_j), \qquad \alpha_i \in \mathbb{C}, \quad m_i \in \mathbb{N}$$

can be solved along the same lines as in Section 4.1 but now using the samples

$$f(t_{\tau+j\sigma}) \sin(\operatorname{Arccos}\ t_{\tau+j\sigma})$$

instead of the $f_{\tau+j\sigma}$, for the sparse interpolation of

$$\sum_{i=1}^{n} \alpha_i \sin\left((m_i+1)\theta_j\right) = \sin\theta_j f(t_j), \qquad t_j = \cos\theta_j.$$

In a very similar way, the sparse interpolation problems

$$f(t_j) = \sum_{i=1}^{n} \alpha_i V_{m_i}(t_j), \qquad \alpha_i \in \mathbb{C}, \quad m_i \in \mathbb{N},$$

$$f(t_j) = \sum_{i=1}^{n} \alpha_i W_{m_i}(t_j), \qquad \alpha_i \in \mathbb{C}, \quad m_i \in \mathbb{N}$$

can be solved, using the Chebyshev polynomials $V_{m_i}(t)$ and $W_{m_i}(t)$ of the third and fourth kind respectively, given by

$$V_m(t) = \frac{\cos\left((n + 1/2)\theta\right)}{\cos(\theta/2)}, \qquad t = \cos\theta, \quad -1 < t \le 1,$$
$$W_m(t) = \frac{\sin\left((n + 1/2)\theta\right)}{\sin(\theta/2)}, \qquad t = \cos\theta, \quad -1 \le t < 1.$$

4.3 Spread polynomials

Let $g(m_i; t)$ equal the degree m_i spread polynomial $S_{m_i}(t)$ on [0, 1], which is defined by

$$S_m(t) = \sin^2(m\theta), \qquad t = \sin^2(\theta), \quad 0 \le t \le 1.$$

The spread polynomials $S_m(t)$ are related to the Chebyshev polynomials of the first kind by $1 - 2t S_m(t) = T_m(1 - 2t)$ and satisfy the recurrence relation

$$S_{m+1}(t) = 2(1-2t)S_m(t) - S_{m-1}(t) + 2t, \qquad S_1(t) = t, \quad S_0(t) = 0$$

and the property

$$S_m(t)S_r(t) = 1/2 S_m(t) + 1/2 S_r(t) - 1/4 S_{m+r}(t) - 1/4 S_{m-r}(t). \tag{29}$$

We consider the interpolation problem

$$f(t_j) = \sum_{i=1}^{n} \alpha_i S_{m_i}(t_j), \qquad \alpha_i \in \mathbb{C}, \quad m_i \in \mathbb{N},$$

where $t_j = \sin^2(j\Delta)$, $j = 0, 1, 2, \ldots$ with $0 < \Delta \le \pi/(2M)$ and $0 < m_1 < \ldots < m_n < M$. The $S_{m_i}(t) = \sin^2(m_i \arcsin\sqrt{t})$ satisfy

$$\frac{1}{2}\left(S_{m_i}(\sin^2\Delta) + S_{m_i}(t_j)\right) - \frac{1}{4}\left(S_{m_i}(t_{j+1}) + S_{m_i}(t_{j-1})\right) = S_{m_i}(\sin^2\Delta)S_{m_i}(t_j).$$

As in Section 4.1 we present a two-tier approach, which for $\sigma \le 1$ reduces to one step and avoids the additional evaluations required for the second step. However, as indicated above, the two-tier scheme offers some additional possibilities. We denote

$$f_{\tau+j\sigma} := \sum_{i=1}^{n} \alpha_i S_{m_i}\left(\sin^2((\tau + j\sigma)\Delta)\right) = \sum_{i=1}^{n} \alpha_i S_{m_i}\left(S_{\tau+j\sigma}(\sin^2 \Delta)\right).$$

With

$$F_{\tau+j\sigma} := F(\sigma, \tau; t_j) = \frac{1}{2}\left(f_\tau + f_{j\sigma}\right) - \frac{1}{4}\left(f_{\tau+j\sigma} + f_{\tau-j\sigma}\right)$$

we obtain

$$F_{\tau+j\sigma} = \sum_{i=1}^{n} \alpha_i S_{m_i}(\sin^2 \tau\Delta) S_{m_i}(\sin^2 j\sigma\Delta).$$

So the effect of the scale factor σ on the one hand and the shift term τ on the other can again be separated in the evaluation $F_{\tau+j\sigma}$.

We introduce the matrices

$$\begin{aligned} {}_{\sigma}J_n &:= \left(1/2 f_{k\sigma} + 1/2 f_{\ell\sigma} - 1/4 f_{(k+\ell)\sigma} - 1/4 f_{(k-\ell)\sigma}\right)_{k,\ell=1}^{n}, \\ {}_{\sigma}^{\tau}K_n &:= \left(1/2 F_{\tau+k\sigma} + 1/2 F_{\tau+\ell\sigma} - 1/4 F_{\tau+(k+\ell)\sigma} - 1/4 F_{\tau+(k-\ell)\sigma}\right)_{k,\ell=1}^{n}. \end{aligned} \tag{30}$$

Theorem 3 *The matrices ${}_{\sigma}^{\tau}K_n$ and ${}_{\sigma}J_n$ factorize as*

$$\begin{aligned} {}_{\sigma}^{\tau}K_n &= R_n L_n A_n R_n^T, \\ {}_{\sigma}J_n &= R_n A_n R_n^T, \\ R_n &= \begin{pmatrix} S_{m_1}(\sin^2 \sigma\Delta) & \cdots & S_{m_n}(\sin^2 \sigma\Delta) \\ \vdots & & \vdots \\ S_{m_1}(\sin^2 n\sigma\Delta) & \cdots & S_{m_n}(\sin^2 n\sigma\Delta) \end{pmatrix}, \\ A_n &= \mathrm{diag}(\alpha_1, \ldots, \alpha_n), \\ L_n &= \mathrm{diag}\left(S_{m_1}(\sin^2 \tau\Delta), \ldots, S_{m_n}(\sin^2 \tau\Delta)\right) \\ &= \mathrm{diag}\left(\sin^2(m_1\tau\Delta), \ldots, \sin^2(m_n\tau\Delta)\right). \end{aligned}$$

Proof The factorization is again verified at the level of the matrix entries, now making use of property (29), which is slightly more particular. □

This factorization paves the way to obtaining the values $S_{m_i}(\sin^2 \sigma\Delta) = \sin^2(m_i\sigma\Delta)$ as the generalized eigenvalues of

$$\left({}_{\sigma}^{\sigma}K_n\right) v_i = S_{m_i}(\sin^2 \sigma\Delta)\,({}_{\sigma}J_n)\, v_i, \qquad i = 1, \ldots, n.$$

Filling the matrices in this matrix pencil requires $2n + 1$ evaluations $f(j\sigma\Delta)$ for $j = 1 \ldots, 2n+1$. From these generalized eigenvalues we cannot necessarily uniquely deduce the values for the indices m_i. Instead, we can obtain for each $i = 1, \ldots, n$ the set of elements

$$S_i = \left(\left\{\frac{\mathrm{Arcsin}(|\sin(m_i\sigma\Delta)|)}{\sigma\Delta} + \frac{\pi}{\sigma\Delta}\ell,\ \ell = 0,\dots\lceil\sigma/2\rceil - 1\right\} \cup \left\{\frac{-\mathrm{Arcsin}(|\sin(m_i\sigma\Delta)|)}{\sigma\Delta} + \frac{\pi}{\sigma\Delta}\ell,\ \ell = 1,\dots\lfloor\sigma/2\rfloor\right\}\right) \cap \mathbb{Z}_M$$

characterizing all the possible values for m_i consistent with the sparse spread polynomial interpolation problem. Fortunately, with $\gcd(\sigma,\tau) = 1$, we can proceed as follows.

First, the coefficients α_i are obtained from the linear system of interpolation conditions

$$\sum_{i=1}^{n} \alpha_i S_{m_i}(\sin^2 j\sigma\Delta) = f_{j\sigma}, \qquad j = 0,\dots,2n-1.$$

The additional values $F_{\tau+j\sigma}$ lead to a second system of interpolation conditions,

$$\sum_{i=1}^{n} \left(\alpha_i S_{m_i}(\sin^2 \tau\Delta)\right) S_{m_i}(\sin^2 j\sigma\Delta) = F_{\tau+j\sigma}, \qquad j = k,\dots,k+n-1, \quad 1 \le k,$$

which delivers the coefficients $\alpha_i S_{m_i}(\sin^2 \tau\Delta)$. Dividing the two solution vectors of these linear systems componentwise delivers the values $S_{m_i}(\sin^2 \tau\Delta)$, $i = 1,\dots,n$ from which we obtain sets

$$T_i = \left(\left\{\frac{\mathrm{Arcsin}(|\sin(m_i\tau\Delta)|)}{\tau\Delta} + \frac{\pi}{\tau\Delta}\ell,\ \ell = 0,\dots\lceil\tau/2\rceil - 1\right\} \cup \left\{\frac{-\mathrm{Arcsin}(|\sin(m_i\tau\Delta)|)}{\tau\Delta} + \frac{\pi}{\tau\Delta}\ell,\ \ell = 1,\dots\lfloor\tau/2\rfloor\right\}\right) \cap \mathbb{Z}_M$$

that have the correct m_i in their intersection with the respective S_i. The proof of this statement follows a completely similar course as that for the cosine building block $g(\phi_i; t)$, given in the Appendix.

The factorization in Theorem 3 allows to write down a spread polynomial analogue of (11).

Corollary 3 *For the matrices* ${}_\sigma J_n$ *and* ${}^0_\sigma K_n$ *defined in* (30) *holds that*

$$\begin{aligned} rank\, {}_\sigma J_\nu &= n, \qquad \nu \ge n,\\ rank\, {}^0_\sigma K_\nu &= n, \qquad \nu \ge n. \end{aligned}$$

To round up the discussion we mention that from Theorem 3 and the generalized eigenvalue problem

$$\left({}^\tau_\sigma K_n\right) v_i = S_{m_i}(\sin^2 \tau\Delta)\, ({}_\sigma J_n)\, v_i, \qquad i = 1,\dots,n,$$

we also find that ${}_\sigma J_n v_i$ is a scalar multiple of

$$\alpha_i \left(S_{m_i}(\sin^2 \sigma\Delta), \ldots, S_{m_i}(\sin^2 n\sigma\Delta) \right)^T .$$

At the expense of some additional samples this eigenvalue and eigenvector combination offers again an alternative computational scheme.

5 Distribution functions

In [29, pp. 85–91] Prony's method is generalized from $g(\phi_i; t) = \exp(\phi_i t)$ with $\phi_i \in \mathbb{C}$ to $g(\phi_i; t) = \exp(-(t - \phi_i)^2)$, to solve the interpolation problem

$$f(t_j) = \sum_{i=1}^{n} \alpha_i \exp\left(-\frac{(t_j - \phi_i)^2}{2w^2}\right), \qquad \alpha_i, \phi_i \in \mathbb{C},$$

with given fixed Gaussian peak width w. Here we further generalize the algorithm to include the new scale and shift paradigm. The scheme is useful when modelling phenomena using Gaussian functions, as illustrated in Section 7.1. Without loss of generality we put $2w^2 = 1$. The easy adaptation to include a fixed constant width factor in the formulas is left to the reader.

We again assume that (4) holds, but now for 2Δ. With $t_j = j\Delta$, the Gaussian $g(\phi_i; t) = \exp(-(t - \phi_i)^2)$ satisfies

$$\exp\left(t_{j+1}^2\right) \exp\left(-(t_{j+1} - \phi_i)^2\right) = \exp(2\phi_i\Delta) \exp\left(t_j^2\right) \exp\left(-(t_j - \phi_i)^2\right).$$

Let us take a closer look at the evaluation of $f(t)$ at $t_{\tau+j\sigma} = (\tau + j\sigma)\Delta$, $j = 0, 1, \ldots$ with $\sigma \in \mathbb{N}$ and $\tau \in \mathbb{Z}$:

$$\exp\left(-((\tau + j\sigma)\Delta - \phi_i)^2\right) = \exp\left(-(\tau\Delta - \phi_i)^2 - j^2\sigma^2\Delta^2 - 2(\tau\Delta - \phi_i)j\sigma\Delta\right).$$

With the auxiliary function

$$\begin{aligned} F(\sigma, \tau; t_j) &:= \exp(2\tau j\sigma\Delta^2) \exp(j^2\sigma^2\Delta^2) f(t_{\tau+j\sigma}) \\ &= \sum_{i=1}^{n} \left(\alpha_i \exp(-(\tau\Delta - \phi_i)^2)\right) \exp(2\phi_i j\sigma\Delta), \end{aligned} \tag{31}$$

we obtain a perfect separation of σ and τ and the problem can be solved using Prony's method. With fixed chosen σ and τ, the value $F(\sigma, \tau; t_j)$ is denoted by $F_{\tau+j\sigma}$.

Theorem 4 *The Hankel structured matrix*

$$ {}_\sigma^\tau G_n := \begin{pmatrix} F_\tau & F_{\tau+\sigma} & \cdots & F_{\tau+(n-1)\sigma} \\ F_{\tau+\sigma} & & & \\ \vdots & & \iddots & \vdots \\ F_{\tau+(n-1)\sigma} & \cdots & & F_{\tau+(2n-2)\sigma} \end{pmatrix} $$

factorizes as

$$ \begin{aligned} {}_\sigma^\tau G_n &= E_n L_n A_n E_n^T, \\ E_n &= \begin{pmatrix} 1 & \cdots & 1 \\ \exp(2\phi_1\sigma\Delta) & \cdots & \exp(2\phi_n\sigma\Delta) \\ \vdots & & \vdots \\ \exp(2\phi_1(n-1)\sigma\Delta) & \cdots & \exp(2\phi_n(n-1)\sigma\Delta) \end{pmatrix} \\ A_n &= \operatorname{diag}(\alpha_1\exp(-\phi_1^2), \ldots, \alpha_n\exp(-\phi_n^2)) \\ L_n &= \operatorname{diag}\left(\exp(-\tau^2\Delta^2 + 2\tau\Delta\phi_1), \ldots, \exp(-\tau^2\Delta^2 + 2\tau\Delta\phi_n)\right). \end{aligned} $$

Proof The proof is again by verification of the entry $F_{\tau+(k+\ell)\sigma}$ in ${}_\sigma^\tau G_n$ at position $(k+1, \ell+1)$ for $k = 0, \ldots, n-1$ and $\ell = 0, \ldots, n-1$. □

With $\tau = 0, \sigma$ the values $\exp(2\phi_i\sigma\Delta)$ are retrieved as a factor of the generalized eigenvalues of the problem

$$ \left({}_\sigma^\sigma G_n\right) v_i = \exp(-\sigma^2\Delta^2)\exp(2\phi_i\sigma\Delta)\left({}_\sigma^0 G_n\right) v_i, \qquad i = 1, \ldots, n. $$

As we know from the exponential case, the ϕ_i cannot necessarily be identified unambiguously from $\exp(2\phi_i\sigma\Delta)$ when $\sigma > 1$. In order to remedy that, we turn our attention to two structured linear systems. The first one, where $\tau = 0$,

$$ \sum_{i=1}^n \alpha_i \exp\left(-(t_{j\sigma} - \phi_i)^2\right) = f_{j\sigma}, \qquad j = 0, \ldots, 2n-1, $$

delivers the $\alpha_i \exp(-\phi_i^2)$ after rewriting it as

$$ \sum_{i=1}^n \left(\alpha_i \exp(-\phi_i^2)\right) \exp(2\phi_i j\sigma\Delta) = \exp(j^2\sigma^2\Delta^2) f_{j\sigma} = F(\sigma, 0; t_j), \qquad j = 0, \ldots, 2n-1. $$

The coefficient matrix of this linear system is Vandermonde structured with entry $(\exp(2\phi_i\sigma\Delta))^j$ at position $(j+1, i)$. The second linear system, where $\tau > 0$, delivers the $\alpha_i \exp(-(\tau\Delta - \phi_i)^2)$ through (31),

$$\sum_{i=1}^{n} \left(\alpha_i \exp(-(\tau\Delta - \phi_i)^2)\right) \exp(2\phi_i j\sigma\Delta) = F_{\tau+j\sigma}, \qquad j = k, \ldots, k+n-1.$$

Here the coefficient matrix is structured identically as in the first linear system. From both solutions we obtain

$$\exp(\tau^2\Delta^2)\frac{\alpha_i \exp(-(\tau\Delta - \phi_i)^2)}{\alpha_i \exp(-\phi_i^2)} = \exp(\tau^2\Delta^2)\frac{\alpha_i \exp(-\tau^2\Delta^2)\exp(-\phi_i^2)\exp(2\phi_i\tau\Delta)}{\alpha_i \exp(-\phi_i^2)} = \exp(2\phi_i\tau\Delta).$$

From the values $\exp(2\phi_i\sigma\Delta), i = 1, \ldots, n$ and $\exp(2\phi_i\tau\Delta), i = 1, \ldots, n$ the parameters $2\phi_i$ can be extracted as explained in Section 2, under the condition that $\gcd(\sigma, \tau) = 1$.

The values $\exp(2\phi_i\tau\Delta)$ and $\exp(2\phi_i\sigma\Delta)$ can also be retrieved respectively from the generalized eigenvalues and the generalized eigenvectors of the alternative problem

$$\left({}_{\sigma}^{\tau}G_n\right) v_i = \exp(-\tau^2\Delta^2)\exp(2\phi_i\tau\Delta)\left({}_{\sigma}^{0}G_n\right) v_i, \qquad i = 1, \ldots, n,$$

with ${}_{\sigma}^{0}G_n v_i$ being a scalar multiple of

$$\alpha_i \left(1, \exp(2\phi_i\sigma\Delta), \ldots, \exp(2\phi_i(n-1)\sigma\Delta)\right)^T,$$

thereby requiring at least of $4n - 2$ samples instead of $3n$ samples. To conclude, the following analogue of (11) can be given.

Corollary 4 *For the matrix* ${}_{\sigma}^{0}G_n$ *given in Theorem 4 holds that*

$$\text{rank}\, {}_{\sigma}^{0}G_\nu = n, \qquad \nu \geq n.$$

6 Some special functions

The sinc function is widely used in digital signal processing, especially in seismic data processing where it is a natural interpolant. There are several similarities between the narrowing sinc function and the Dirac delta function, among which the shape of the pulse. A large number of papers, among which [30], already discuss the determination of a so-called train of Dirac spikes and their amplitudes, which is essentially an exponential fitting problem. This is generalized here to the use of the sinc function, including a matrix pencil formulation.

The gamma function first arose in connection with the interpolation problem of finding a function that equals $n!$ when the argument is a positive integer. Nowadays the function plays an important role in mathematics, physics and engineering. In [10] sparse interpolation or exponential analysis was already generalized to the Pochhammer basis $(t)_m = t(t+1)\cdots(t+m-1)$, also called rising factorial, which is related to the gamma function by $(t)_m = \Gamma(t+m)/\Gamma(t)$ for $t \notin \mathbb{Z}^- \cup \{0\}$. Here we generalize the method to the direct use of the gamma function and we present a matrix pencil formulation as well.

6.1 The sampling function $\sin(x)/x$

Let $g(\phi_i; t) = \mathrm{sinc}(\phi_i t)$ where $\mathrm{sinc}(t)$ is historically defined by $\mathrm{sinc}(t) = \sin t/t$. So our sparse interpolation problem is

$$f(t_j) = \sum_{i=1}^{n} \alpha_i \,\mathrm{sinc}(\phi_i t_j), \qquad t_j = j\Delta,$$

with the same assumptions for ϕ_i and Δ as in Section 3. In order to solve this inverse problem of identifying the ϕ_i and α_i for $i = 1, \ldots, n$, we introduce

$$F(t_j) := j\Delta f(t_j) = \sum_{i=1}^{n} \left(\frac{\alpha_i}{\phi_i}\right) \sin(\phi_i j\Delta)$$

and apply the technique from Section 3.2 for the separate identification of the nonlinear parameters ϕ_i and linear parameters α_i/ϕ_i in the sparse sine interpolation.

6.2 The gamma function $\Gamma(z)$

With the new tools obtained so far, it is also possible to extend the theory to other functions such as the gamma function $\Gamma(z)$. The function $g(\phi_i; z) = \Gamma(z+\phi_i)$ with $z, \phi_i \in \mathbb{C}$, satisfies the relation

$$\Gamma(\Delta+1+\phi_i) = (\Delta+\phi_i)\Gamma(\Delta+\phi_i), \qquad \Delta \in \mathbb{C},\ \Delta+\phi_i \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}. \quad (32)$$

Our interest is in the sparse interpolation of

$$f(z) = \sum_{i=1}^{n} \alpha_i \Gamma(z+\phi_i), \qquad z+\phi_i \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$$

where the $\alpha_i, \phi_i, i = 1, \ldots, n$ are unknown. In the sample point $z = \Delta$ we define

$$\begin{aligned} F_0(\Delta) &:= f(\Delta), \\ F_j(\Delta) &:= F_{j-1}(\Delta+1) - \Delta F_{j-1}(\Delta), \qquad j = 1, 2, \ldots \end{aligned}$$

If by the choice of Δ, one or more of the $\Delta+\phi_i, i = 1, \ldots, n$ accidentally belong to the set of nonpositive integers, then one cannot sample $f(z)$ at $z = \Delta$. In that case a complex shift τ can help out. It suffices to shift the arguments $\Delta + \phi_i$ away from the negative real axis. We then redefine

$$F_{\tau,j}(\Delta) := F_j(\tau + \Delta), \qquad \tau \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$$

or in other words

$$\begin{aligned} F_{\tau,0}(\Delta) &:= f(\tau + \Delta), \\ F_{\tau,j}(\Delta) &:= F_{\tau,j-1}(\Delta + 1) - (\tau + \Delta)F_{\tau,j-1}(\Delta), \qquad j = 1, 2, \ldots \end{aligned}$$

Using (32) we find

$$F_{\tau,j}(\Delta) = \sum_{i=1}^{n} \alpha_i\, \phi_i^j\; \Gamma(\tau + \Delta + \phi_i), \qquad j = 0, 1, 2, \ldots$$

If $\tau = 0$ then $F_{0,j} = F_j(\Delta)$. As soon as the samples at $\tau + \Delta + j$ are all well-defined, we can start the algorithm for the computation of the unknown linear parameters α_i and the nonlinear parameters ϕ_i, We further introduce

$$ {}_1^{\tau,k}\mathcal{H}_n := \begin{pmatrix} F_{\tau,k} & \cdots & F_{\tau,k+n-1} \\ \vdots & \iddots & \vdots \\ F_{\tau,k+n-1} & \cdots & F_{\tau,k+2n-2} \end{pmatrix}.$$

Theorem 5 *The matrix ${}_1^{\tau,k}\mathcal{H}_n$ is factored as*

$$\begin{aligned} {}_1^{\tau,k}\mathcal{H}_n &= \mathcal{P}_n P_n Z_n \mathcal{P}_n^T, \\ \mathcal{P}_n &= \begin{pmatrix} 1 & \cdots & 1 \\ \phi_1 & \cdots & \phi_n \\ \vdots & & \vdots \\ \phi_1^{n-1} & \cdots & \phi_n^{n-1} \end{pmatrix}, \\ Z_n &= \mathrm{diag}\left(\alpha_1\Gamma(\tau + \Delta + \phi_1), \ldots, \alpha_n\Gamma(\tau + \Delta + \phi_n)\right), \\ P_n &= \mathrm{diag}(\phi_1^k, \ldots, \phi_n^k). \end{aligned}$$

Proof With the matrix factorization given, the proof consists of an easy verification of the matrix product with the matrix ${}_1^{\tau,k}\mathcal{H}_n$. □

Filling the matrices ${}_1^{\tau,0}\mathcal{H}_n$ and ${}_1^{\tau,1}\mathcal{H}_n$ requires the evaluation of $f(z)$ at $z = \tau + \Delta + j$, $j = 0, \ldots, 2n-1$ which are points on a straight line parallel with the real axis in the complex plane.

The nonlinear parameters ϕ_i are now obtained as the generalized eigenvalues of

$$\left({}_1^{\tau,1}\mathcal{H}_n\right) v_i = \phi_i \left({}_1^{\tau,0}\mathcal{H}_n\right) v_i, \qquad i = 1, \ldots, n, \tag{33}$$

where the v_i, $i = 1, \ldots, n$ are the right generalized eigenvectors. Afterwards the linear parameters α_i are obtained from the linear system of interpolation conditions

$$\sum_{i=1}^{n} (\alpha_i \Gamma(\tau + \Delta + \phi_i))\, \phi_i^j = F_{\tau,j}(\Delta), \qquad j = \tau, \ldots, \tau + 2n - 1,$$

by computing the coefficients $\alpha_i \Gamma(\tau + \Delta + \phi_i)$ and dividing those by the function values $\Gamma(\tau + \Delta + \phi_i)$ which are known because Δ, τ and the ϕ_i, $i = 1, \ldots, n$ are known.

From Theorem 5 we find that for the generalized eigenvectors of (33) holds that ${}_1^{\tau,0}\mathcal{H}_n v_i$ is a scalar multiple of

$$\alpha_i \left(1, \phi_i, \ldots, \phi_i^{n-1}\right)^T.$$

This allows to validate the computation of the ϕ_i, $i = 1, \ldots, n$ obtained as generalized eigenvalues, if desired.

6.3 Pochhammer basis connection

Results on sparse polynomial interpolation using the Pochhammer basis $(t)_m$ where usually the interpolation points are positive integers and $t \in \mathbb{R}^+$, were published in [10, 28], but no matrix pencil method for its solution was presented. This can now easily be obtained using a similar approach as for the gamma function. We consider more generally the interpolation of

$$f(z) = \sum_{i=1}^{n} \alpha_i \ (z)_{m_i}, \qquad z \in \mathbb{C}\backslash\{0\}, \quad m_i \in \mathbb{N}.$$

For complex values z, the Pochhammer basis or rising factorial $(z)_m$ satisfies the recurrence relation

$$z \ [(z+1)_m - (z)_m] = m \ (z)_m.$$

For real Δ, a complex shift τ could shift the problem statement away from the negative real axis, as with the gamma function, but it is much simpler here to immediately consider $\Delta \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$. Let

$$F_0 := F_0(\Delta) = f(\Delta),$$

$$F_j := F_j(\Delta) = \Delta \ \left[F_{j-1}(\Delta + 1) - F_{j-1}(\Delta)\right] = \sum_{i=1}^{n} \alpha_i m_i^j \ (\Delta)_{m_i}, \qquad j = 1, 2, \ldots$$

With the evaluations F_j we fill the Hankel matrix

$$ {}_1^k H_n = \begin{pmatrix} F_k & F_{k+1} & \cdots & F_{k+n-1} \\ F_{k+1} & & & \\ \vdots & & \iddots & \vdots \\ F_{k+n-1} & \cdots & & F_{k+2n-1} \end{pmatrix}. $$

This Hankel matrix decomposes as in Theorem 5, but now with

$$ \mathcal{P}_n = \begin{pmatrix} 1 & \cdots & 1 \\ m_1 & \cdots & m_n \\ \vdots & & \vdots \\ m_1^{n-1} & \cdots & m_n^{n-1} \end{pmatrix}, $$

$$ Z_n = \operatorname{diag}\left(\alpha_1 \ (\Delta)_{m_1}, \ldots, \alpha_n \ (\Delta)_{m_n}\right), $$

$$ P_n = \operatorname{diag}(m_1^k, \ldots, m_n^k). $$

So the nonlinear parameters $m_i, i = 1, \ldots, n$ are obtained as the generalized eigenvalues of

$$ {}_1^1 H_n v_i = m_i \ {}_1^0 H_n v_i, \qquad i = 1, \ldots, n $$

where the v_i are the right generalized eigenvectors, for which holds that ${}_1^0 H_n v_i$ is a multiple of the vector

$$ \alpha_i \ (\Delta)_{m_i} \left(1, m_i, \ldots, m_i^{n-1}\right)^T. $$

From the latter the estimates of the m_i can be validated by computing the quotient of successive entries in the vector. The linear parameters $\alpha_i, i = 1, \ldots, n$ are obtained from the linear system

$$ \sum_{i=1}^{n} \alpha_i m_i^j \ (\Delta)_{m_i} = F_j, \qquad j = 0, \ldots, 2n-1. $$

7 Numerical illustrations

We present some examples to illustrate the main novelties of the paper, including the multiscale facilities:

- an illustration of sparse interpolation by Gaussian distributions with fixed width but unknown peak locations;
- an illustration of the new generalized eigenvalue formulation for use with several trigonometric functions and the sinc;
- an illustration of the use of the scale and shift strategy for the supersparse interpolation of polynomials.

As stated earlier, our focus is on the mathematical generalizations and not on the numerical issues.

7.1 Fixed width sparse Gaussian fitting

Consider the expression

$$f(t) = \exp(-(t-5)^2) + 0.01\exp(-(t-4.99)^2),$$

illustrated in Fig. 1, with the parameters $\alpha_i, \phi_i \in \mathbb{R}$. From the plot it is not obvious that the signal has two peaks.

We first show the output of the widely used [31] Matlab state-of-the-art peak fitting program `peakfit.m`, which calls an unconstrained nonlinear optimization algorithm to decompose an overlapping peak signal into its components [32, 33].

In `peakfit.m` the user needs to supply a guess for the number of peaks and supply this as input. If one does not have any idea on the number of peaks, the usual practice is to try different possibilities and compare the corresponding results. Of course, a good estimate of the number of peaks may lead to a good fit of the data. In addition to the peak position ϕ_i, its height α_i and width w, the program also returns a goodness-of-fit (`GOF`).

The `peakfit.m` algorithm can work without assuming a fixed width, or the width can be passed as an argument. We do the latter as our algorithm also assumes a known fixed peak width w.

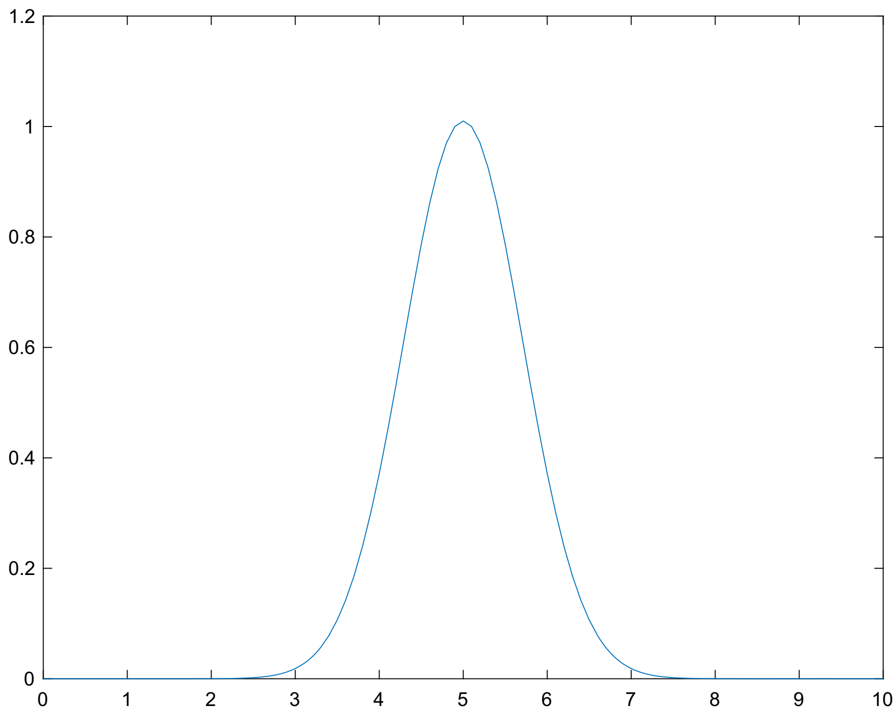

Fig. 1 Plot of Gaussian example function

Let $\Delta = 0.1$ and let us collect 20 samples $f_0, f_1, \ldots, f_{19}$. When passing the width to `peakfit.m` and guessing the number of peaks, then it returns for 1 peak the estimates

$$\phi_1 = 4.9998944950\ldots \qquad \alpha_1 = 1.0099771180\ldots \qquad \mathtt{G}OF \approx 1.5 \times 10^{-5}.$$

For 2 peaks it returns

$$\begin{aligned} \phi_1 &= 4.9998944843\ldots & \alpha_1 &= 1.0099776250\ldots \\ \phi_2 &= -1.5242538810\ldots & \alpha_2 &= 1.05 \times 10^{-13} \end{aligned} \qquad \mathtt{G}OF \approx 5.2 \times 10^{-7}.$$

Since the result is still not matching our benchmark input parameters, let us push further and supply 100 samples. Then for 1 peak `peakfit.m` returns

$$\phi_1 = 4.9999009752\ldots \qquad \alpha_1 = 1.0099995049\ldots \qquad \mathtt{G}OF \approx 2.5 \times 10^{-5}$$

and for 2 peaks we get

$$\begin{aligned} \phi_1 &= 4.9999945211\ldots & \alpha_1 &= 1.0010660101\ldots \\ \phi_2 &= 4.9894206737\ldots & \alpha_2 &= 0.0089339897\ldots \end{aligned} \qquad \mathtt{G}OF \approx 8.4 \times 10^{-9}.$$

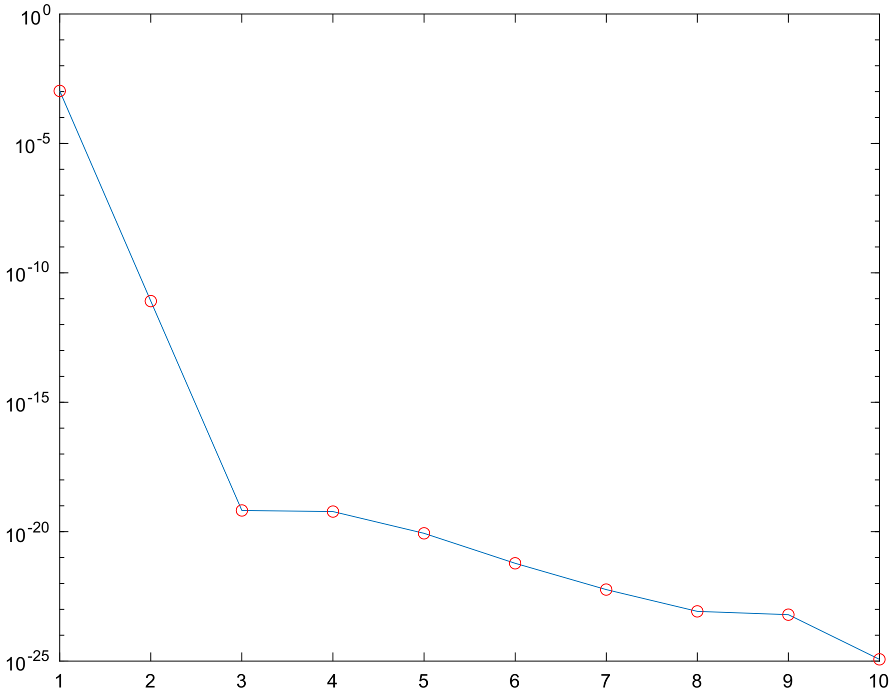

Fig. 2 Singular value log-plot of the matrix ${}^{0}_{1}G_{10}$

From the latter experiment it is easy to formulate some desired features for a new algorithm:

- built-in guess of the number of peaks in the signal,
- and reliable output from a smaller number of samples.

So let us investigate the technique developed in Section 5. Take $\sigma = 1$ and $\tau = 0$ since there is no periodic component in the Gaussian signal, which has only real parameters. With the 20 samples $f_0, f_1, \dots, f_{19}$ we define the samples $F_j = \exp(j^2\Delta^2) f_j$ and compose the Hankel matrix ${}^0_1G_{10}$. Its singular value decomposition, illustrated in Fig. 2, clearly reveals that the rank of the matrix is 2 and so we deduce that there are $n = 2$ peaks.

From the 4 samples F_0, F_1, F_2, F_3 we obtain through Theorem 7

$$\phi_1 = 4.9999999737\dots \qquad \phi_2 = 4.9899976207\dots$$
$$\alpha_1 = 1.0000049866\dots \qquad \alpha_2 = 0.0099950129\dots$$

The new method clearly provides both an automatic guess of the number of peaks and a reliable estimate of the signal parameters, all from only 20 samples. What remains to be done is to investigate the numerical behaviour of the method on a large collection of different input signals, which falls out of the scope of this paper where we provide the mathematical details.

7.2 Sparse sinc interpolation

Consider the function

$$f(t) = -10\mathrm{sinc}(145.5t) + 20\mathrm{sinc}(149t) + 4\mathrm{sinc}(147.3t),$$

plotted in Fig. 3, which we sample at $t_j = j\pi/300$ for $j = 0, \dots, 19$. The singular value decomposition of ${}^0_1B_{10}$ filled with the values $t_j f_j$, of which the log-plot is shown in Fig. 4 (left), reveals that $f(t)$ consists of 3 terms. Remember that the sparse sinc interpolation problem with linear coefficients α_i and nonlinear parameters ϕ_i transforms into a sparse sine interpolation problem with linear coefficients α_i/ϕ_i and samples $j\Delta f_j$.

The condition numbers of the matrices ${}^0_1B_3$ and ${}^1_1B_3$ appearing in the generalized eigenvalue problem

$$\left({}^1_1B_3\right) v_i = \cos(\phi_i \Delta) \left({}^0_1B_3\right), \qquad i = 1, 2, 3$$

equal respectively 1.6×10^7 and 7.5×10^6. To improve the conditioning of the structured matrix we choose $\sigma = 30$, $\tau = 1$ and resample $f(t)$ at $t_j = 30j\pi/300 = j\pi/10$ for $j = 0, \dots, 5$. The singular values of ${}^0_\sigma B_{10}$ are graphed in Fig. 4 (right) and the condition numbers of ${}^0_\sigma B_3$ and ${}^1_\sigma B_3$ improve to 1.1×10^3 and 9.7×10^2 respectively.

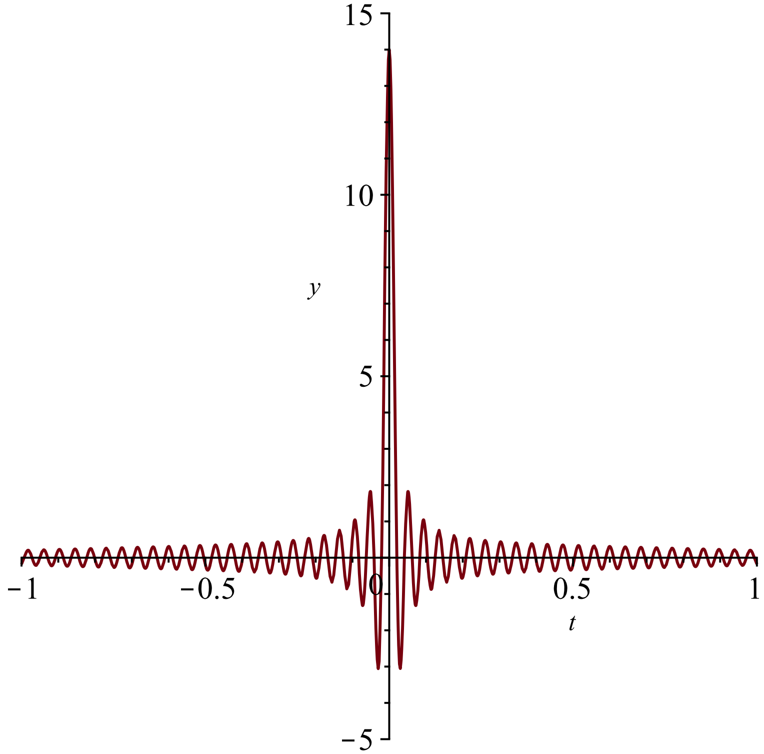

Fig. 3 The 3-term sparse sinc expression $f(t)$

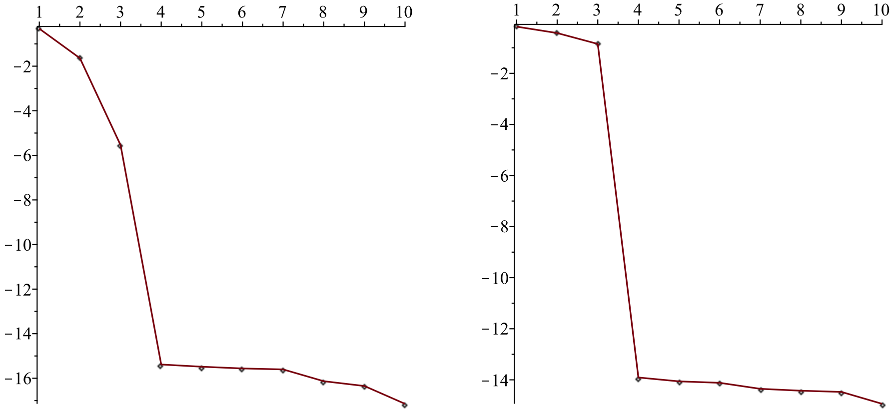

Fig. 4 Singular value log-plot of ${}_{1}^{0}B_{10}$ (left) and ${}_{30}^{0}B_{10}$ (right)

The generalized eigenvalues of the matrix pencil ${}^1_\sigma B_3 - \lambda^0_\sigma B_3$ are given by

$$\cos(30\phi_1\Delta) = -0.1564344650400536,$$
$$\cos(30\phi_2\Delta) = -0.9510565162957546,$$
$$\cos(30\phi_3\Delta) = -0.6613118653271576$$

and with these we fill the matrix W_3 from Theorem 2. We solve (23) for the values $\alpha_i \sin(\phi_i\sigma\Delta)/\phi_i$, $i = 1, 2, 3$ and further compute for $j = 1, \ldots, n$,

$$\frac{\alpha_i}{\phi_i}\sin(\phi_i j\sigma\Delta) = \frac{\alpha_i}{\phi_i}\sin(\phi_i(j-1)\sigma\Delta)\cos(\phi_i\sigma\Delta) + \cos(\phi_i(j-1)\sigma\Delta)\frac{\alpha_i}{\phi_i}\sin(\phi_i\sigma\Delta).$$

At this point the matrix U_3 from Theorem 2 can be filled and the $\cos(\phi_i\tau\Delta)$ can be computed from (24), with the right-hand side filled with the additional samples $F(31\Delta)$, $F(61\Delta)$, $F(91\Delta)$, where

$$F_{\tau+j\sigma} = \frac{\Delta}{2}(\tau + j\sigma) f_{\tau+j\sigma} + \frac{\Delta}{2}(-\tau + j\sigma) f_{-\tau+j\sigma}.$$

Since $\tau = 1$ we obtain the ϕ_i directly from the values $\cos(\phi_i\tau\Delta)$: $\phi_1 = 145.5000000000$, $\phi_2 = 149.0000000000$, $\phi_3 = 147.3000000000$. The linear coefficients α_i are given by

$$\alpha_i = \phi_i \frac{\alpha_i \sin(\phi_i\sigma\Delta)/\phi_i}{\sin(\phi_i\sigma\Delta)},$$

resulting in $\alpha_1 = -9.999999999991$, $\alpha_2 = 19.99999999978$, $\alpha_3 = 4.000000000089$.

7.3 Supersparse Chebyshev interpolation

We consider the polynomial

$$f(t) = 2T_6(t) + T_7(t) + T_{39999}(t),$$

which is clearly supersparse when expressed in the Chebyshev basis. We sample $f(t)$ at $t_j = \cos(j\Delta)$ where $\Delta = \pi/100000$ with $M = 50000$. The first challenge is maybe to retrieve an indication of the sparsity n.

Take $\sigma = 1$ and collect 15 samples f_j, $j = 0, \ldots, 14$ to form the matrix ${}^0_1C_8$. From its singular value decomposition, computed in double precision arithmetic and illustrated on the log-plot in Fig. 5, one may erroneously conclude that $f(t)$ has only 2 terms, a consequence of the fact that, relatively speaking, the degrees $m_1 = 6$ and $m_2 = 7$ are close to one another and appear as one cluster.

Imposing a 3-term model to $f(t)$ instead of the erroneously suggested 2-term one, does not improve the computation as the matrix ${}^0_1C_8$ is ill-conditioned with a condition number of the order of 10^{10}. So 3 generalized eigenvalues cannot be extracted reliably from the samples. For completeness we mention the unreliable double precision results, rounded to integer values: $m_1 = 6$, $m_2 = 39999$, $m_3 = 25119$.

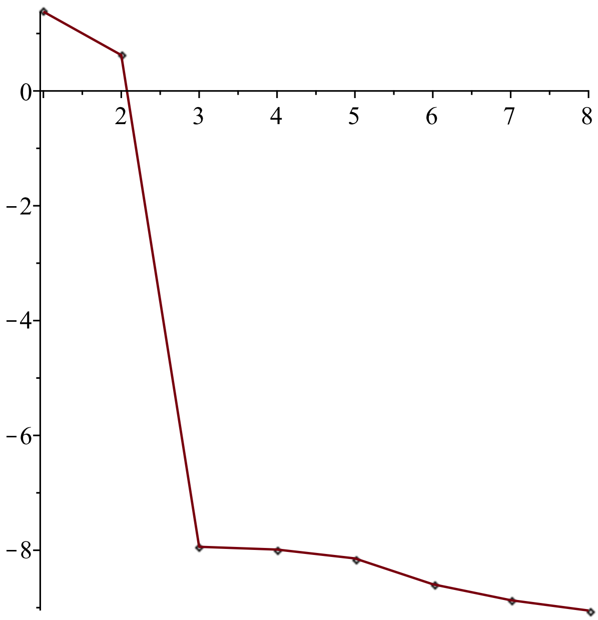

Fig. 5 Log-plot of singular values of ${}^{0}_{1}C_8$

Now choose $\sigma = 3125$ and $\tau = 16$. The singular value decomposition of ${}^{0}_{3125}C_8$, shown on the log-plot in Fig. 6, reveals that $f(t)$ indeed consists of 3 terms. Also, the conditioning of the involved matrix ${}^{0}_{3125}C_8$ has improved to the order of 10^3.

The distinct generalized eigenvalues extracted from the matrix pencil ${}^{1}_{3125}C_3 - \lambda {}^{0}_{3125}C_3$ are given by

$$\cos(m_1\sigma\Delta) = 0.9999999204093383,$$
$$\cos(m_2\sigma\Delta) = -0.8089800617792506,$$
$$\cos(m_3\sigma\Delta) = -0.007490918959382487.$$

From 3 shifted samples at the arguments $t_{\tau+j\sigma}$, $j = 0, 1, 2$ we obtain

$$\cos(m_1\tau\Delta) = -0.8084256802389809,$$
$$\cos(m_2\tau\Delta) = 0.9999752362021560,$$
$$\cos(m_3\tau\Delta) = 0.9999818099296417.$$

Building the sets S_i and T_i for $i = 1, 2, 3$ as indicated in Section 4.1, and rounding the result to the nearest integer, does unfortunately not provide singletons for $S_1 \cap T_1$, $S_2 \cap T_2$, $S_3 \cap T_3$. We consequently need to consider a second shift, for which we choose $\sigma + \tau = 3141$. With this choice we only need to add the evaluation of $f(\tau + n\sigma)$ to

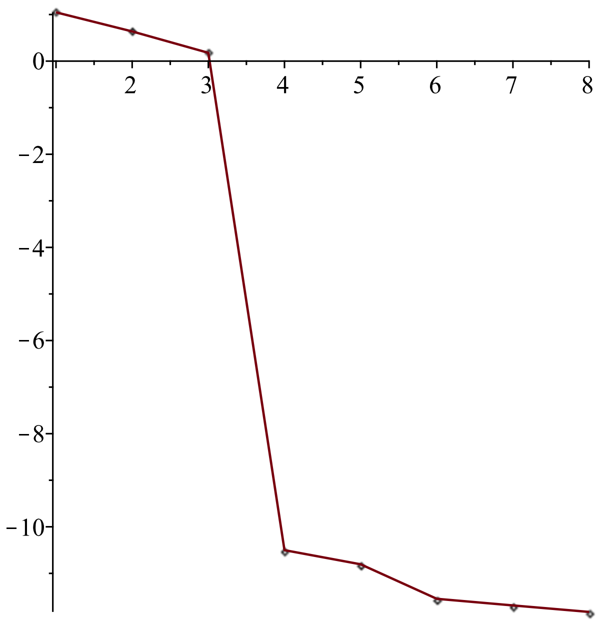

Fig. 6 Log-plot of singular values of ${}^{0}_{3125}C_8$

proceed and compute

$$\cos(m_1(\sigma+\tau)\Delta) = -0.6780621808989576,$$
$$\cos(m_2(\sigma+\tau)\Delta) = 0.1881836009619241,$$
$$\cos(m_3(\sigma+\tau)\Delta) = 0.3771037932233129.$$

Finally, intersecting each $S_i \cap T_i$, $i = 1, 2, 3$ with the solutions provided by the second shift, delivers the correct $m_1 = 6$, $m_2 = 7$, $m_3 = 39999$.

8 Conclusion

Let us summarize the sparse interpolation formulas obtained in the preceding sections in a table. For each parameterized univariate function $g(\phi_i; t)$ we list in the columns 1 to 4:

1. the minimal number of samples required to solve the sparse interpolation without running into ambiguity problems, meaning for the choice $\sigma = 1$,
2. the minimal number of samples required for the choice $\sigma > 1$ (if applicable), thereby involving a shift $\tau \neq 0$ to restore uniqueness of the solution,
3. the linear matrix pencil (A, B) in the generalized eigenvalue formulation $Av_i = \lambda_i Bv_i$ of the sparse interpolation problem involving the $g(\phi_i; t)$,
4. the generalized eigenvalues in terms of τ, as they can be read directly from the structured matrix factorizations presented in the theorems 1–5,

5. the information that can be computed from the associated generalized eigenvectors, as indicated at the end of each (sub)section.

$g(\phi_i; t)$	# samples		pencil* (A, B)	λ_i	Bv_i
	$\sigma = 1$	$\sigma > 1$			
$\exp(\phi_i t)$	$2n$	$3n$	$\left({}^{\tau}_{\sigma}H_n, {}^{0}_{\sigma}H_n\right)$	$\exp(\phi_i \tau \Delta)$	$\alpha_i, \exp(\phi_i \sigma \Delta)$
$\cos(\phi_i t)$	$2n$	$4n$	$\left({}^{\tau}_{\sigma}C_n, {}^{0}_{\sigma}C_n\right)$	$\cos(\phi_i \tau \Delta)$	$\alpha_i, \cos(\phi_i \sigma \Delta)$
$\sin(\phi_i t)$	$2n$	$4n + 2$	$\left({}^{\tau}_{\sigma}B_n, {}^{0}_{\sigma}B_n\right)$	$\cos(\phi_i \tau \Delta)$	$\alpha_i, \sin(\phi_i \sigma \Delta)$
$\cosh(\phi_i t)$	$2n$	$4n$	$\left({}^{\tau}_{\sigma}C_n^*, {}^{0}_{\sigma}C_n^*\right)$	$\cosh(\phi_i \tau \Delta)$	$\alpha_i, \cosh(\phi_i \sigma \Delta)$
$\sinh(\phi_i t)$	$2n$	$4n + 2$	$\left({}^{\tau}_{\sigma}B_n^*, {}^{0}_{\sigma}B_n^*\right)$	$\cosh(\phi_i \tau \Delta)$	$\alpha_i, \sinh(\phi_i \sigma \Delta)$
$T_{m_i}(t)$	$2n$	$4n$	$\left({}^{\tau}_{\sigma}C_n, {}^{0}_{\sigma}C_n\right)$	$T_{m_i}(\cos \tau \Delta)$	$\alpha_i, T_{m_i}(\cos \sigma \Delta)$
$S_{m_i}(t)$	$2n + 1$	$4n + 2$	$\left({}^{\tau}_{\sigma}K_n, \sigma J_n\right)$	$S_{m_i}(\sin^2 \tau \Delta)$	$\alpha_i, S_{m_i}(\sin^2 \sigma \Delta)$
$\mathrm{sinc}(\phi_i t)$	$2n$	$4n + 2$	$\left({}^{\tau}_{\sigma}B_n, {}^{0}_{\sigma}B_n\right)$	$\cos(\phi_i \tau \Delta)$	$\alpha_i, \sin(\phi_i \sigma \Delta)$
$\Gamma(z + \phi_i)$	$2n$	x	$\left({}^{\tau,1}_{\sigma}\mathcal{H}_n, {}^{\tau,0}_{\sigma}\mathcal{H}_n\right)$	ϕ_i	α_i, ϕ_i
$\exp(-(t - \phi_i)^2)$	$2n$	$3n$	$\left({}^{\tau}_{\sigma}G_n, {}^{0}_{\sigma}G_n\right)$	$\exp(2\phi_i \tau \Delta)$	$\alpha_i, \exp(2\phi_i \sigma \Delta)$

*With $\cos(\cdot)$ replaced by $\cosh(\cdot)$ and $\sin(\cdot)$ replaced by $\sinh(\cdot)$ in the hyperbolic case

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Availability of data and materials Code generating the data and running all examples will be available from the website http://cemath.org.

Declarations

Ethics approval Not applicable

Conflict of interest The authors declare no competing interests.

Appendix

To reconstruct a function of the form

$$f(t) = \sum_{i=1}^{n} \alpha_i \cos(\phi_i t)$$

from equidistantly collected samples f_j at $t = j\Delta$, in other words to recover the unknown parameters ϕ_i and coefficients α_i, the sampling step Δ needs to satisfy the Shannon-Nyquist constraint

$$\Delta < \pi / \max_{i=1,\ldots,n} |\phi_i|.$$

Since we do not distinguish ϕ_i from $-\phi_i$ in this case, we can simply drop the sign information in ϕ_i from here on and write $\Delta = \pi/R$ with

$$0 \le \max_{i=1,\ldots,n} \phi_i < R.$$

The challenge we consider now is to retrieve the parameters ϕ_i and coefficients α_i from sub-Nyquist rate collected samples $f_{j\sigma}$ at $t = j\sigma\Delta$ with $\sigma > 1$ and the shifted evaluations $f_{j\sigma+\tau}$ at $t = (j\sigma + \tau)\Delta$ with $\gcd(\sigma, \tau) = 1$. In Section 3.1 we describe how for $i = 1, \ldots, n$ the values

$$\begin{aligned} C_{i,\sigma} &:= \cos(\phi_i \sigma \Delta), \\ C_{i,\tau} &:= \cos(\phi_i \tau \Delta) \end{aligned}$$

are obtained. The aim is to extract the correct value for ϕ_i from the knowledge of the evaluations $C_{i,\sigma}$ and $C_{i,\tau}$, particularly when $(\sigma\Delta) \max_{i=1,\ldots,n} \phi_i \ge \pi$ and the parameter ϕ_i cannot be obtained uniquely from $C_{i,\sigma}$ alone. We now discuss the unique identification of this parameter ϕ_i and in doing so we further drop the index i. Let us denote

$$\begin{aligned} A_\sigma &:= \operatorname{Arccos}(C_\sigma), \\ A_\tau &:= \operatorname{Arccos}(C_\tau) \end{aligned} \tag{34}$$

where $\operatorname{Arccos}(\cdot) \in [0, \pi]$ indicates the principal value of the inverse cosine function. Knowing that $0 \le A_\sigma, A_\tau \le \pi$ and that $0 \le \phi\sigma\Delta < \sigma\pi$, we find that all possible positive arguments $\phi\sigma\Delta$ of C_σ are in $\mathcal{A}_{\sigma,1} \cup \mathcal{A}_{\sigma,2}$ with

$$\begin{aligned} \mathcal{A}_{\sigma,1} &:= \{A_\sigma + 2\pi\ell \mid 0 \le \ell \le \lceil \sigma/2 \rceil - 1\}, \\ \mathcal{A}_{\sigma,2} &:= \{(2\pi - A_\sigma)\operatorname{sgn}(A_\sigma) + 2\pi\ell \mid 0 \le \ell \le \lceil \sigma/2 \rceil - 1\}, \end{aligned}$$

where $\operatorname{sgn}(a_\sigma) = +1$ for $0 < A_\sigma \le \pi$ and $\operatorname{sgn}(0) = 0$. The set $\mathcal{A}_{\sigma,1} \cup \mathcal{A}_{\sigma,2}$ may even contain some candidate arguments of C_σ that do not satisfy the bounds, but this does not create a problem in the identification of the correct $\phi < R$. Along the same lines, sets $\mathcal{A}_{\tau,1}$ and $\mathcal{A}_{\tau,2}$ can be constructed.

We further denote

$$\phi_\sigma := A_\sigma/(\sigma\Delta) = \frac{A_\sigma R}{\sigma\pi},$$
$$\phi_\tau := A_\tau/(\tau\Delta) = \frac{A_\tau R}{\tau\pi}. \tag{35}$$

Then the possible solutions for ϕ to $C_\sigma = \cos(\phi\sigma\Delta)$ are in $\Phi_{\sigma,1} \cup \Phi_{\sigma,2}$ where

$$\Phi_{\sigma,1} := \{\phi_\sigma + 2R\ell/\sigma \mid 0 \le \ell \le \lceil\sigma/2\rceil - 1\} \cap [0, R), \tag{36}$$
$$\Phi_{\sigma,2} := \{(2R/\sigma - \phi_\sigma)\,\mathrm{sgn}(\phi_\sigma) + 2R\ell/\sigma \mid 0 \le \ell \le \lceil\sigma/2\rceil - 1\} \cap [0, R). \tag{37}$$

Analogously, the possible solutions to $C_\tau = \cos(\phi\tau\Delta)$ are in $\Phi_{\tau,1} \cup \Phi_{\tau,2}$ where

$$\Phi_{\tau,1} := \{\phi_\tau + 2R\ell/\tau \mid 0 \le \ell \le \lceil\tau/2\rceil - 1\} \cap [0, R), \tag{38}$$
$$\Phi_{\tau,2} := \{(2R/\tau - \phi_\tau)\,\mathrm{sgn}(\phi_\tau) + 2R\ell/\tau \mid 0 \le \ell \le \lceil\tau/2\rceil - 1\} \cap [0, R). \tag{39}$$

One statement is obvious: whatever the choice for σ and τ, both $\Phi_{\sigma,1} \cup \Phi_{\sigma,2}$ and $\Phi_{\tau,1} \cup \Phi_{\tau,2}$ contain the unknown value for ϕ which produced C_σ and C_τ. What remains open is the question whether $\left(\Phi_{\sigma,1} \cup \Phi_{\sigma,2}\right) \cap \left(\Phi_{\sigma,1} \cup \Phi_{\sigma,2}\right)$ is a singleton. And in case it is not, we want to find an algorithm that can identify the correct ϕ.

When either $\phi_\sigma = 0$ or $\phi_\sigma = R/\sigma$ the sets $\Phi_{\sigma,1}$ and $\Phi_{\sigma,2}$ coincide. And similarly for ϕ_τ. On the other hand, if these sets do not coincide, they are disjoint. So the true value for the unknown parameter ϕ can belong to any of the intersections $\Phi_{\sigma,1} \cap \Phi_{\tau,1}$, $\Phi_{\sigma,1} \cap \Phi_{\tau,2}$, $\Phi_{\sigma,2} \cap \Phi_{\tau,1}$, $\Phi_{\sigma,2} \cap \Phi_{\tau,2}$. A sequence of lemmas will lead to the conclusion that the four intersections do not deliver more than two distinct elements. Thereafter we indicate how to identify the only true value for the unkown ϕ.

Lemma 1 $i, j \in \{1, 2\} : \Phi_{\sigma,i} \cap \Phi_{\tau,j} \neq \emptyset \Longrightarrow \#\left(\Phi_{\sigma,i} \cap \Phi_{\tau,j}\right) = 1.$

Proof Without loss of generality we prove the statement for $i = 1 = j$, by contraposition. The proof of the other cases is entirely similar. From $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \emptyset$ and containing at least two elements, we then find that

$$\exists 0 \le \ell_1, \ell_2 \le \lceil\sigma/2\rceil - 1, 0 \le k_1, k_2 \le \lceil\tau/2\rceil - 1, \ell_1 \neq \ell_2, k_1 \neq k_2 :$$
$$\begin{cases} \phi_\sigma + \ell_1 2R/\sigma = \phi_\tau + k_1 2R/\tau \\ \phi_\sigma + \ell_2 2R/\sigma = \phi_\tau + k_2 2R/\tau. \end{cases}$$

This leads to

$$\frac{\ell_1 - \ell_2}{k_1 - k_2} = \frac{\sigma}{\tau}$$

which is a contradiction because $|\ell_1 - \ell_2| < \sigma$, $|k_1 - k_2| < \tau$ and $\gcd(\sigma, \tau) = 1$. □

When the sets $\Phi_{\sigma,1}$ and $\Phi_{\sigma,2}$ coincide and the sets $\Phi_{\tau,1}$ and $\Phi_{\tau,2}$ do as well, then that unique intersection is $\phi = \phi_\sigma = \phi_\tau = 0$. Because $\gcd(\sigma, \tau) = 1$, other common elements coming from either $\phi_\sigma = 2R/\sigma$ or $\phi_\tau = 2R/\tau$ cannot exist.

We now continue with the situation where either the sets in (36) and (37) or the sets in (38) and (39) do not coincide, so that there are always at least 3 distinct sets in the running. Without loss of generality, we assume that a common element belongs to $\Phi_{\sigma,1} \cap \Phi_{\tau,1}$ and we build our reasoning from there.

Lemma 2 $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \emptyset \Longrightarrow \Phi_{\sigma,2} \cap \Phi_{\tau,2} = \emptyset$.

Proof We know that either $\Phi_{\sigma,1} \cap \Phi_{\sigma,2} \neq \emptyset$ or $\Phi_{\tau,1} \cap \Phi_{\tau,2} \neq \emptyset$ and possibly both, so that $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \Phi_{\sigma,2} \cap \Phi_{\tau,2}$. Again by contraposition, we suppose that $\Phi_{\sigma,2} \cap \Phi_{\tau,2} \neq \emptyset$ and so

$$\exists 0 \le \ell_1, \ell_2 \le \lceil \sigma/2 \rceil - 1, 0 \le k_1, k_2 \le \lceil \tau/2 \rceil - 1 :$$
$$\begin{cases} \phi_1 = \phi_\sigma + \ell_1 2R/\sigma = \phi_\tau + k_1 2R/\tau \in \Phi_{\sigma,1} \cap \Phi_{\tau,1}, \\ \phi_2 = 2R/\sigma - \phi_\sigma + \ell_2 2R/\sigma = 2R/\tau - \phi_\tau + k_2 2R/\tau \in \Phi_{\sigma,2} \cap \Phi_{\tau,2}. \end{cases}$$

From this we obtain
$$\frac{1+\ell_1+\ell_2}{1+k_1+k_2} = \frac{\sigma}{\tau}$$
which can only be true when $1 + \ell_1 + \ell_2 = \sigma$ and $1 + k_1 + k_2 = \tau$. Then

$$\phi_1 + \phi_2 = (1 + \ell_1 + \ell_2) 2R/\sigma = 2R,$$

which contradicts $0 \le \phi_1, \phi_2 < R$. □

While, assuming $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \emptyset$, we have seen in Lemma 1 that this intersection is a singleton, and we have seen in Lemma 2 that then $\Phi_{\sigma,2} \cap \Phi_{\tau,2} = \emptyset$, we know nothing so far about the other two intersections $\Phi_{\sigma,1} \cap \Phi_{\tau,2}$ and $\Phi_{\sigma,2} \cap \Phi_{\tau,1}$.

Lemma 3 $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \emptyset \Longrightarrow \neg \left(\Phi_{\sigma,2} \cap \Phi_{\tau,1} \neq \emptyset \wedge \Phi_{\sigma,1} \cap \Phi_{\tau,2} \neq \emptyset \right)$.

Proof By contraposition we assume that

$$\exists 0 \le \ell_1, \ell_2 \le \lceil \sigma/2 \rceil - 1, 0 \le k_1, k_2 \le \lceil \tau/2 \rceil - 1 :$$
$$\begin{cases} \phi_1 = 2R/\sigma - \phi_\sigma + \ell_1 2R/\sigma = \phi_\tau + k_1 2R/\tau \in \Phi_{\sigma,2} \cap \Phi_{\tau_1}, \\ \phi_2 = \phi_\sigma + \ell_2 2R/\sigma = 2R/\tau - \phi_\tau + k_2 2R/\tau \in \Phi_{\sigma,1} \cap \Phi_{\tau,2}. \end{cases}$$

This leads to
$$\frac{1+\ell_1+\ell_2}{1+k_1+k_2} = \frac{\sigma}{\tau},$$
which again implies $1 + \ell_1 + \ell_2 = \sigma$ and $1 + k_1 + k_2 = \tau$. Since

$$\phi_1 + \phi_2 = (1 + \ell_1 + \ell_2) 2R/\sigma = 2R,$$

this contradicts $0 \le \phi_1, \phi_2 < R$. □

We have built our sequence of proofs from Lemma 2 on, without loss of generality, on the fact that $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \emptyset$ and the fact that $\Phi_{\sigma,1}$ and $\Phi_{\sigma,2}$ on the one hand and $\Phi_{\tau,1}$ and $\Phi_{\tau,2}$ on the other do not collide at the same time. Finally, from Lemma 3 we know that $(\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \emptyset$ and $\Phi_{\sigma,2} \cap \Phi_{\tau,1} \neq \emptyset)$ or $(\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \emptyset$ and $\Phi_{\sigma,1} \cap \Phi_{\tau,2} \neq \emptyset)$ cannot occur concurrently, but either one of these cases remains possible.

In general, when at least 3 of the 4 sets $\Phi_{\sigma,1}, \Phi_{\sigma,2}, \Phi_{\tau,1}, \Phi_{\tau,2}$ are distinct, then at most 2 of the 4 intersections

$$\Phi_{\sigma,i} \cap \Phi_{\tau,j}, \qquad 1 \leq i, j \leq 2$$

are nonempty, with each of the nonempty intersections being a singleton. Further down we illustrate the actual existence of a case, where two intersections are nonempty and consequently the true value of the unknown ϕ cannot be identified from the evaluations $\cos(\phi\sigma\Delta)$ and $\cos(\phi\tau\Delta)$ with $\gcd(\sigma, \tau) = 1$.

In this case we need to collect a third value $C_\rho := \cos(\phi\rho\Delta)$ with $\gcd(\sigma, \rho) = 1$ and $\gcd(\tau, \rho) = 1$. With A_ρ and ϕ_ρ defined as in (34) and (35), and $\Phi_{\rho,1}$ and $\Phi_{\rho,2}$ defined as in (36) and (37), we know, as before, that $\Phi_{\rho,1} \cup \Phi_{\rho,2}$ contains the correct value for ϕ. We also know, because of the remark formulated after the proof of Lemma 1, that at least 5 of the 6 involved sets $\Phi_{\sigma,1}, \Phi_{\sigma,2}, \Phi_{\tau,1}, \Phi_{\tau,2}, \Phi_{\rho,1}, \Phi_{\rho,2}$ are distinct unless $\phi = 0$.

We now inspect

$$\begin{aligned} \left[\cup_{i,j=1}^{2} \left(\Phi_{\sigma,i} \cap \Phi_{\tau,j}\right)\right] \cap \left(\Phi_{\rho,1} \cup \Phi_{\rho,2}\right) \\ = \left(\cup_{k=1}^{2} \Phi_{\sigma,i_1} \cap \Phi_{\tau,j_1} \cap \Phi_{\rho,k}\right) \\ \cup \left(\cup_{k=1}^{2} \Phi_{\sigma,i_2} \cap \Phi_{\tau,j_2} \cap \Phi_{\rho,k}\right) \end{aligned} \tag{40}$$

where i_1, j_1, i_2, j_2 index the subsets that produce the nonempty intersections of the relatively prime pair σ and τ, with either $i_1 \neq i_2$ or $j_1 \neq j_2$ but not both. We have built our sequence of proofs, without loss of generality, on the fact that $i_1 = 1, j_1 = 1$ and have found that it is then possible that $i_2 = 2, j_2 = 1$. We now continue the proofs from that case and inspect the 4 new intersections in (40).

Lemma 4 $\Phi_{\rho,1} \cap \Phi_{\rho,2} = \emptyset \wedge \Phi_{\sigma,1} \cap \Phi_{\tau,1} \cap \Phi_{\rho,1} \neq \emptyset \Longrightarrow \Phi_{\sigma,1} \cap \Phi_{\tau,1} \cap \Phi_{\rho,2} = \emptyset.$

Proof From Lemma 1, we know that $\Phi_{\sigma,1} \cap \Phi_{\tau,1}$ is a singleton. If that unique element also belongs to $\Phi_{\rho,1}$ then it cannot belong to $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \cap \Phi_{\rho,2}$ when $\Phi_{\rho,1}$ and $\Phi_{\rho,2}$ are disjoint. □

Lemma 5 $\Phi_{\sigma,1} \cap \Phi_{\sigma,2} = \emptyset \wedge \Phi_{\sigma,1} \cap \Phi_{\tau,1} \cap \Phi_{\rho,1} \neq \emptyset \Longrightarrow \Phi_{\sigma,2} \cap \Phi_{\tau,1} \cap \Phi_{\rho,1} = \emptyset.$

Proof From Lemma 1, we know that $\Phi_{\tau,1} \cap \Phi_{\rho,1}$ is a singleton. If that unique element also belongs to $\Phi_{\sigma,1}$ then it cannot belong to $\Phi_{\sigma,2} \cap \Phi_{\tau,1} \cap \Phi_{\rho,1}$ when $\Phi_{\sigma,1}$ and $\Phi_{\sigma,2}$ are disjoint. □

As a consequence of the Lemmas 4 and 5, the unique true ϕ is identified in

$$\Phi_{\sigma,1} \cap \Phi_{\tau,1} \cap \Phi_{\rho,1} \text{ or } \Phi_{\sigma,2} \cap \Phi_{\tau,1} \cap \Phi_{\rho,2}.$$

Lemma 6 $\#\left[\left(\Phi_{\sigma,1} \cap \Phi_{\tau,1} \cap \Phi_{\rho,1}\right) \cup \left(\Phi_{\sigma,2} \cap \Phi_{\tau,1} \cap \Phi_{\rho,2}\right)\right] = 1.$

Proof We know that either $\phi = 0$ is the unique element in the intersections or at least 2 of the intersections $\Phi_{\sigma,1} \cap \Phi_{\sigma,2}$, $\Phi_{\tau,1} \cap \Phi_{\tau,2}$, $\Phi_{\rho,1} \cap \Phi_{\rho,2}$ are empty. So either $\Phi_{\sigma,1} \cap \Phi_{\sigma,2} = \emptyset$ or $\Phi_{\rho,1} \cap \Phi_{\rho,2} = \emptyset$. When applying Lemma 2 to the pair (σ, ρ) instead of (σ, τ) the set $\Phi_{\sigma,2} \cap \Phi_{\rho,2} = \emptyset$ if the set $\Phi_{\sigma,1} \cap \Phi_{\rho,1} \neq \emptyset$. Therefore two distinct elements in respectively $\Phi_{\sigma,1} \cap \Phi_{\rho,1}$ and $\Phi_{\sigma,2} \cap \Phi_{\rho,2}$ cannot coexist and solve $C_\tau = \cos(\phi\tau\Delta)$. □

So the unknown parameter ϕ is identified uniquely from at most 3 values C_σ, C_τ, C_ρ with σ, τ, ρ all mutually prime. An easy choice for ρ is $\rho = \sigma + \tau$ as this minimizes the number of additional samples as explained in Section 3, and also $\gcd(\sigma, \sigma + \tau) = 1 = \gcd(\tau, \sigma + \tau)$ when $\gcd(\sigma, \tau) = 1$.

As promised, we show an example where $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \emptyset$ and $\Phi_{\sigma,2} \cap \Phi_{\tau,1} \neq \emptyset$. Consider $\phi = 70800/1547 < 1000 = R$ with $\Delta = \pi/R$. Choose $\sigma = 299$ and $\tau = 357$ with $\gcd(\sigma, \tau) = 1$. With

$$\phi_\sigma = 100000/35581, \qquad \ell = 68 \leq \lceil \sigma/2 \rceil - 1 = 149,$$

we have $\phi \in \Phi_{\sigma,1}$. With

$$\phi_\tau = 6000/1547, \qquad \ell = 81 \leq \lceil \tau/2 \rceil - 1 = 178,$$

we find $\phi \in \Phi_{\tau,1}$. Unfortunately, since $\phi_\tau = 2R/\sigma - \phi_\sigma$ we also have $\phi_\tau \in \Phi_{\sigma,2} \cap \Phi_{\tau,1} \neq \emptyset$.

As a last remark, we add that even replacing (13) by the stricter constraint

$$|\phi_i|\Delta < \pi/2, \qquad i = 1, \ldots, n$$

does not guarantee that each ϕ can be identified from only C_σ and C_τ. We illustrate this with a counterexample. Let $\phi = 3300/133 < 50 = R$ with $\Delta = \pi/(2R)$. With $\sigma = 21$ and $\tau = 19$ we find

$$\begin{aligned} \phi_\sigma &= 500/133, \qquad (2\pi)/(\sigma\Delta) - \phi_\sigma = 2300/399, \\ \phi_\tau &= 500/133, \qquad (2\pi)/(\tau\Delta) - \phi_\tau = 900/133. \end{aligned}$$

This leads to $\Phi_{\sigma,1} \cap \Phi_{\tau,1} = \{500/133\}$ and $\Phi_{\sigma,2} \cap \Phi_{\tau,1} = \{3300/133\}$.

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