Report from Dagstuhl Seminar 22221

Exponential Analysis: Theoretical Progress and Technological Innovation

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Abstract

Multi-exponential analysis might sound remote, but it touches our daily lives in many surprising ways, even if most people are unaware of how important it is. For example, a substantial amount of effort in signal processing and time series analysis is essentially dedicated to the analysis of multi-exponential functions. Multi- exponential analysis is also fundamental to several research fields and application domains that have been the subject of this Dagstuhl seminar: remote sensing, antenna design, digital imaging, all impacting some major societal or industrial challenges such as energy, transportation, space research, health and telecommunications. This Seminar connected stakeholders from seemingly separately developed fields: computational harmonic analysis, numerical linear algebra, computer algebra, nonlinear approximation theory, digital signal processing and their applications, in one and more variables.

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1 Executive Summary

Annie Cuyt (University of Antwerp, BE)
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For the analysis and representation of stationary signals and images the conventional Fourier- and wavelet-based methods are particularly appropriate. However, in many areas in science and engineering we are faced with the problem to interpret digital signals and images which are not band-limited and have a non-stationary behaviour. Frequently, there are even further obstacles. The acquisition of signal or image measurements may be very expensive and therefore limited. In other applications, measurement sets are huge but contaminated by noise. Examples of the above are encountered in magnetic resonance imaging, infrared microscopy, fluorescence-lifetime imaging microscopy (FLIM), the analysis of seismic signals

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in geophysics, radar imaging (SAR/ISAR), tissue ageing models, vibration analysis, direction of arrival (DOA) detection, texture classification, radio frequency identification (RFID), non-destructive testing, satellite navigation, time series analysis, echolocation, induction motor diagnostics (MCSA), to name just a few.

Within the last few years, research on Prony-based methods has been intensified, as they offer an alternative to the compressed sensing approach. One essential advantage of the Prony method is that it does not need randomly collected measurements but works with deterministic sampling based on a sampling scheme which is adapted to the nonlinear signal model. At the same time, the Prony approach does not suffer the well-known curse of dimensionality in the multivariate case.

This Dagstuhl Seminar brought together a number of researchers from different areas in mathematics, engineering and industry. Topics included new mathematical insights and efficient numerical algorithms for problems based on exponential analysis as well as applications of exponential analysis models in engineering and life sciences. During this Seminar, the participants presented their newest results and discussed several open problems and applications from different perspectives.

The talks in the workshop partially had the character of survey talks and were arranged in different main research topics, as new mathematical theory (Day 1 and Day 5), new computational approaches (Day 2) and new results in applications in engineering (Day 3) and life sciences (Day 4).

On the first day, the talks focussed on the **mathematical theory of exponential analysis**. B. Beckermann and A. Matos emphasized the close connection between exponential analysis and rational approximation, which leads to new application areas as density reconstruction in an equilibrium problem in logarithmic potential theory and improved rational approximation of Markov functions. As shown in the talk by T. Sauer, there is also a close connection between multi-exponential analysis and continued fractions.

In the afternoon, the participants started four smaller **thematic discussion groups** to discuss new challenges and open problems throughout the week. These discussion groups particularly focussed on theory, more efficient computational algorithms and applications in engineering and biosciences. The discussions included general observations on the further development of exponential analysis and its ties to other subjects as well as further specific approaches for application of the theory to application problems in radar imaging or MRI. Still, we are far away from a full understanding of the relations between the different methods for the stable reconstruction of a parametric signal model, as well as from a systematic construction of Fourier analytic methods for the improved analysis of non-stationary signals.

In practice, Prony-based methods often suffer from a bad conditioning of the involved structured matrices and some extra effort is required to reliably execute the corresponding algorithms. Among the more successful implementations, the ESPRIT method, the Matrix-Pencil method, the approximate Prony method, and validated exponential analysis are established. The problem statement is also closely related to rational approximation theory and the structured low-rank approximation of structured matrices. These connections are still not completely understood and may lead to strongly improved reconstruction algorithms, able to treat more general sampling sets and deliver super-resolution results.

The second day was devoted to the problem of **efficient computations** for reconstruction or approximation of functions using multi-exponential analysis. First numerical methods like MUSIC are already implemented in the computer algebra system Maplesoft, as presented in the talk by J. Gerhard. The advantage of the application of Maplesoft is that it can process the data within a desired very high precision and therefore successfully handle these reconstruction problems which are known to be ill-conditioned. Another way to improve the numerical stability is the application of more sophisticated numerical approaches to structured matrices and the connection to rational approximation, such that the existing stable algorithms for rational approximation can be used, see the talk of G. Plonka. The survey presentation by H. Mhaskar turned the attention to the application of neural networks to approximation problems, and D. Potts presented new efficient Fourier methods to compute the so-called ANOVA decomposition for approximation of multivariate functions.

In the afternoon a mini-course on newest features of Maplesoft was presented by J. Gerhard, with a special focus on the problem of how to connect MAPLESOFT with other Software as MATLAB etc. The remaining time was used for further scientific discussions in smaller groups.

On the third day, the talks surveyed different new **applications of exponential analysis in engineering**. The presentation of F. Knaepkens showed new approaches for direction of arrival (DOA) estimation, image denoising and inverse synthetic aperture radar. Chromatic aberration in large antenna systems have been studied by D. de Villiers. R.-M Weideman showed her results on antenna position estimation through sub-sampled exponential analysis of signals in the near-field. The talks by R. Beinert and J. Prestin showed applications concerning phase retrieval in optical diffraction tomography and detection of directional jumps in images.

The survey presentations on the fourth day focussed on **exponential models in life sciences**. J. Gielis showed in his talk, how generalized Möbius-Listing bodies (GML) can be employed for better modelling and understanding of certain dynamical processes in the natural sciences. D. Li explained recent progress of advanced time-resolved imaging techniques based on exponential analysis models and their applications in life sciences, for example to reveal biological processes at the molecular level. In a further talk, R.G. Spencer reported on newest developments in Magnetic Resonance Relaxometry and Macromolecular Mapping to achieve more accurate myelin quantification in the brain that permits the establishment of physiological correlations. The underlying reconstruction problem is a seriously ill-posed inverse problem.

The last day of the workshop was again devoted to further results in **mathematical theory of exponential analysis** and connections to other areas of mathematics. In the talk by D. Batenkow, the degree of ill-posedness of the parameter reconstruction problem based on the exponential sum model was studied in more detail. The degree of condition of the problem essentially depends on the distribution of the frequency parameters. H. Vesolovska discussed the problem of recovering an atomic measure on the unit 2-sphere $\mathbb{S}^2$ given finitely many moments with respect to spherical harmonics. A connection of exponential analysis to computer algebra problems was brought to our attention by M. Ishteva. She showed how the joint decomposition of a set of non-homogeneous polynomials can be computed using the canonical polyadic decomposition, and how this decomposition can be applied in nonlinear system identification.

This Dagstuhl meeting has been an important milestone for improved understanding of the large impact of exponential analysis tools in both theory and practice. During the thematic discussions in small groups in this meeting, several new aspects have been considered and several collaborations have been initiated or continued. Examples include new approaches for an improved modelling of antenna system frequency responses in radio frequency (Cuyt, De Villiers, Weideman) and for stabilised parameter estimation in exponential models using iterative factorizations of matrix pencils of Loewner matrices (Beckermann, Plonka-Hoch).

We mention that this seminar is related to Dagstuhl seminar 15251 on “Sparse modelling and multi-exponential analysis” that took place in 2015. The discussions at the latter have led to many interesting collaborative projects, among which a funded Horizon-2020 RISE project (Research and Innovation Staff Exchange) with the acronym EXPOWER, standing for “Exponential analysis Empowering innovation” (grant agreement No 101008231).

It is our experience that these Dagstuhl seminars are timely and seminal. Through the meetings new collaborations and new potential are unlocked. There is a clear need to further connect stakeholders from the new theoretical developments and the identified industrial applications, as is our objective here and in the future.

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3 Overview of Talks

3.1 Limits of sparse super-resolution

Dmitry Batenkov (Tel Aviv University, IL)

We consider the recovery of parameters $\{c_j, x_j\}$ in exponential sums

$$f(\omega) = \sum_{j=1}^{s} c_j \exp(\imath x_j \omega)$$

from bandlimited and noisy samples

$$f(\omega_i) + \epsilon_i, \qquad \omega_i \in [-\Omega, \Omega], \quad |\epsilon_i| \leq \varepsilon.$$

We discuss the conditioning of the problem when some of the exponents $\{x_i\}$ become close to each other, and show that all model parameters can be stably recovered provided that $\varepsilon \leq c(\Omega\Delta)^{2\ell-1}$, where ℓ is the maximal number of exponents which can be within an interval of size $\approx 1/\Omega$, and Δ is the a-priori minimal separation between the $\{x_j\}$.

We also discuss extensions of the analysis to generalized exponential sums $f(\omega) = \sum_{j=1}^{s}(a_j + b_j\omega)\exp(\imath x_j \omega)$, and connections to spectral properties of (confluent) Vandermonde matrices with nodes on the unit circle.

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1 D. Batenkov, L. Demanet, G. Goldman, and Y. Yomdin, "Conditioning of Partial Nonuniform Fourier Matrices with Clustered Nodes," *SIAM J. Matrix Anal. Appl.*, **44**(1), 199–220, 2020.
2 D. Batenkov, B. Diederichs, G. Goldman, and Y. Yomdin, "The spectral properties of Vandermonde matrices with clustered nodes," *Linear Algebra and its Applications*, **609**, 37–72, 2021.
3 D. Batenkov and G. Goldman, "Single-exponential bounds for the smallest singular value of Vandermonde matrices in the sub-Rayleigh regime," *Applied and Computational Harmonic Analysis*, **55**, 426–439, 2021.
4 D. Batenkov, G. Goldman, and Y. Yomdin, "Super-resolution of near-colliding point sources," *Inf Inference*, **10**(2), 515–572, 2021.
5 D. Batenkov and N. Diab, "Super-resolution of generalized spikes and spectra of confluent Vandermonde matrices," *arXiv:2203.11923*.

3.2 Best rational approximants of Markov functions

Bernhard Beckermann (University of Lille, FR)

Main reference Bernhard Beckermann, Joanna Bisch, Robert Luce: "On the rational approximation of Markov functions, with applications to the computation of Markov functions of Toeplitz matrices", Numer Algor 91, 109–144, (2022).
URL https://doi.org/10.1007/s11075-022-01256-4

The study of the error of rational approximants of Markov functions

$$f^{[\mu]}(z) = \int \frac{d\mu(x)}{z - x}, \quad \text{supp}(\mu) \subset [a, b],$$

on some $E \subset R \setminus [\alpha, \beta]$ has a long history, with a well-established link to orthogonal polynomials. For example, Zolotarev more than 100 years ago described best rational approximants and their error for the particular Markov function

$$f^{[\nu]}(z) = \frac{\sqrt{|a|}}{\sqrt{(z-a)(z-b)}} = \int \frac{d\nu(x)}{z-x}, \quad \frac{d\nu}{dx}(x) = \frac{\sqrt{|a|}}{\pi\sqrt{(x-a)(b-x)}},$$

for closed intervals E. The aim of this talk is to show that

$$\min_{r \in \mathcal{R}_{m-1,m}} \|1 - r/f^{[\mu]}\|_{L^\infty(E)} \leq 3 \min_{r \in \mathcal{R}_{m-1,m}} \|1 - r/f^{[\nu]}\|_{L^\infty(E)},$$

that is, up to some modest factor, the particular Markov function $f^{[\nu]}$ gives the worst relative error among all Markov functions $f^{[\mu]}$. In our proof we show similar inequalities for rational interpolants and Padé approximants.

3.3 Total-Variation-Based Phase Retrieval in Optical Diffraction Tomography

Robert Beinert (TU Berlin, DE)

Joint work of Robert Reinert, Michael Quellmalz
Main reference Robert Beinert, Michael Quellmalz: "Total Variation-Based Reconstruction and Phase Retrieval for Diffraction Tomography", SIAM Journal on Imaging Sciences, Vol. 15(3), pp. 1373–1399, 2022.
URL http://dx.doi.org/10.1137/22M1474382

In optical diffraction tomography (ODT), the three-dimensional scattering potential of a microscopic object rotating around its center is recovered by a series of illuminations with coherent light. Reconstruction algorithms such as the filtered backpropagation require knowledge of the complex-valued wave at the measurement plane, whereas often only intensities, i.e., phaseless measurements, are available in practice.

In this talk, we propose a new reconstruction approach for ODT with unknown phase information based on three key ingredients. First, the light propagation is modeled using Born's approximation enabling us to use the Fourier diffraction theorem. Second, we stabilize the inversion of the non-uniform discrete Fourier transform via total variation regularization utilizing a primal-dual iteration, which also yields a novel numerical inversion formula for ODT with known phase. The third ingredient is a hybrid input-output scheme. We achieved convincing numerical results, which indicate that ODT with phaseless data is possible. The so-obtained 2D and 3D reconstructions are even comparable to the ones with known phase.

3.4 Chromatic Aberration in Large Antenna Systems

Dirk de Villiers (University of Stellenbosch, ZA)

It is well understood in electromagnetic field theory that harmonic plane waves, traveling in the same direction but emanating from different locations, interfere to cause a total field with a complex exponential frequency dependence. This effect often manifests in large (many wavelengths in size) antennas as a frequency ripple in the port reflection coefficient and radiation pattern magnitudes. The mechanism generating the ripple, or chromatic aberration [1], can normally be attributed to either multiple reflections from different structures in the antenna, or due to interference between direct and diffracted waves in the structure [2].

Often the resulting effect is small by virtue of the fundamental design of the structures at hand, but for certain applications even such small effects can influence the efficacy of the total system. Many radio astronomy science cases are examples of such sensitive applications, where the experiment searches for extremely feint signals (often deep in the noise) that need to be accurately characterized as a function of frequency. Here, we need to model our antenna system frequency responses very accurately in order to de-embed them from the data, and the chromatic aberration effects must be accounted for. Antennas typically used for radio astronomy, namely large reflectors or smaller antennas suspended over an artificial or natural ground plane, all suffer from this aberration to some degree. Previous efforts to model this effect are rather crude, and opens the door for more careful consideration and application of exponential analysis to reconstruct the required frequency responses without resorting to extremely time consuming high frequency resolution numerical analyses.

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2 D.I.L. de Villiers, Prediction of Aperture Efficiency Ripple in Clear Aperture Offset Gregorian Antennas. IEEE Transactions on Antennas and Propagation 61(5), 2457-–2465, 2013.
3 D.I.L. de Villiers, Wideband Reflector Antenna Beam Pattern Frequency Variation Modeling through the Characteristic Basis Function Pattern Method. In Int. Conference on Electromagnetics in Advanced Applications (ICEAA), 220–223., Cartagena de Indias, 2018.

3.5 New Features in Maple 2022

Jürgen Gerhard (Maplesoft– Waterloo, CA)

An overview of the new features in Maple 2022 will be given, including formal power series, step-by-step solutions, intersection multiplicities, print layout mode, signal processing, and many more.

3.6 Generalized Möbius-Listing bodies– new models for the sciences

Johan Gielis (NL)

Generalized Möbius-Listing surfaces and bodies (GML) were introduced by Ilia Tavkhelidze [1, 2], building on the idea of Gaspard Monge to understand complex movements as the composition of simple movements. Another motivation is that the solutions of BVP for partial differential equations strongly depend on the topological properties of the domain in which the problem is considered.

A main focus of our joint research has been to classify all possible ways of cutting GML bodies (in analogy with the cutting of Möbius bands) [2, 3]. In general the cutting process yields intertwined and linked bodies or surfaces of complex topology.

We defined the conditions under which the cutting process results in a single surface or body only, displaying the Möbius phenomenon of one-sidedness [4]. At the crossroads of geometry, topology, algebra and number theory, this gives rise to new ways of modeling and understanding certain dynamical processes in the natural sciences.

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2. I. Tavkhelidze, C. Cassisa, P.E. Ricci, About connection of generalized Möbius-Listing surfaces with sets of knots and links. *Lecture notes of Seminario Interdisciplinare di Matematica* **IX**, 183-200 2010.
3. J. Gielis and I. Tavkhelidze, The general case of cutting of GML surfaces and bodies. *4OPEN* **3**(7), 2020.
4. J. Gielis, P.E. Ricci, I. Tavkhelidze, The Möbius phenomenon in Generalized Möbius-Listing surfaces and bodies, and Arnold's Cat phenomenon. *Advanced Studies: Euro-Tbilisi Mathematical Journal* **14**(4), 17-35, 2021.

3.7 A modified Waring's problem: decoupling a multivariate polynomial

Mariya Ishteva (KU Leuven– Geel, BE)

Joint work of Mariya Ishteva, Philippe Dreesen
Main reference Philippe Dreesen, Mariya Ishteva, Johan Schoukens: "Decoupling Multivariate Polynomials Using First-Order Information and Tensor Decompositions", SIAM J. Matrix Anal. Appl., Vol. 36(2), pp. 864–879, 2015.
URL http://dx.doi.org/10.1137/140991546

The Waring's problem for polynomials is a fundamental problem in mathematics, concerning the decomposition of a homogeneous multivariate polynomial $f(u_1, \ldots, u_m)$ of degree d as

$$f(u_1, \ldots, u_m) = \sum_{i=1}^{r} w_i(v_1 u_1 + \cdots + v_m u_m)^d,$$

in which r denotes the so-called Waring rank.

We consider a set of non-homogeneous polynomials and show how their joint decomposition can be computed using the canonical polyadic decomposition, which is a well-studied tensor decomposition. We also mention an application in nonlinear system identification.

3.8 Least-squares Multidimensional Exponential Analysis

Ferre Knaepkens (University of Antwerp, BE)

Joint work of Annie Cuyt, Yuan Hou, Ferre Knaepkens, Wen-shin Lee
Main reference Ferre Knaepkens, Annie Cuyt, Wen-Shin Lee, Dirk I. L. de Villiers: "Regular Sparse Array Direction of Arrival Estimation in One Dimension", IEEE Transactions on Antennas and Propagation, Vol. 68(5), pp. 3997–4006, 2020.
URL http://dx.doi.org/10.1109/TAP.2019.2963618
Main reference Annie Cuyt, Yuan Hou, Ferre Knaepkens, Wen-shin Lee: "Sparse Multidimensional Exponential Analysis with an Application to Radar Imaging", SIAM Journal on Scientific Computing, Vol. 42(3), pp. B675–B695, 2020.
URL http://dx.doi.org/10.1137/19M1278004

Exponential analysis consists in extracting the coefficients α_j, $j = 1, \dots, n$, and exponents ϕ_j, $j = 1, \dots, n$, of an exponential model

$$f(x) = \sum_{j=1}^{n} \alpha_j \exp(\phi_j x),$$

from a limited number of observations of the model's behaviour. Since there are $2n$ unkown parameters for n exponential terms, this directly leads to a non-linear square system of $2n$ equations. However, in practice the signal is often perturbed by noise, hence, additional samples are collected and accumulated in an overdetermined non-linear system. Now the question remains how such a noisy overdetermined system behaves and how we can use this information to improve the accuracy of the results. In the case of a square system it is shown in [1] that the exponential analysis problem is deeply intertwined with Padé approximation theory and symmetric tensor decomposition, for both the one-dimensional and multi-dimensional cases. In particular the connection with Padé approximations is very interesting, since it allows the use of Froissart doublets to effectively filter out the noise and correctly estimate the number of terms n. It still remains to be shown that these properties also hold for the least-squares setting of the exponential analysis problem.

Furthermore, we focus on three different application domains, ranging from only one dimension to three-dimensional problems, each with its own challenges. First up is one-dimensional direction of arrival estimation, then image denoising of structured images and finally inverse synthetic aperture radar. We combine sub-Nyquist sampling, a validation technique based on Froissart doublets and new matrix pencil methods in order to tackle these challenging engineering applications.

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1. F. Knaepkens, A. Cuyt, and W.-s. Lee, From Exponential Analysis to Padé Approximation and Tensor Decomposition, in One and More Dimensions. In *Computer Algebra in Scientific Computing*, G. Vladimir, K. Wolfram, S. Werner, and V. Evgenii (eds.), 116–130. Lille, 2018.

3.9 Advanced time-resolved imaging techniques and analysis and their applications in life sciences

David Li (The University of Strathclyde– Glasgow, GB)

Main reference Dong Xiao, Zhenya Zang, Natakorn Sapermsap, Quan Wang, Wujun Xie, Yu Chen, David Day Uei Li: "Dynamic fluorescence lifetime sensing with CMOS single-photon avalanche diode arrays and deep learning processors", Biomed. Opt. Express, Vol. 12(6), pp. 3450–3462, Optica Publishing Group, 2021.
URL http://dx.doi.org/10.1364/BOE.425663

Advanced time-resolved imaging techniques can reveal biological processes at the molecular level. They can unravel mechanisms underlying diseases and facilitate drug development. However, we also enter the low light regime (only a few photons are acquired), and new acquisition and analysis approaches are sought after.

3.10 Solving a class of singular integral equations using rational approximation

Ana C. Matos (Lille I University, FR)

Joint work of Ana C. Matos, Bernhard Beckermann

We are interested in computing the unknown density of an equilibrium problem in logarithmic potential theory, where the support of the equilibrium measure is a finite union of distinct intervals. We will show that this problem is equivalent to solving a system of singular integral equations with Cauchy kernels. After fixing the functional spaces where we search for the solution , we obtain a theorem of existence and unicity. We then develop a general framework of a spectral method to compute an approximate solution , giving a complete error analysis. We will consider polynomial [1] and rational approximations [2, 3], showing the advantage of using rational interpolation when the intervals are close. Inspired by the third Zolotareff problem, the poles and the interpolation points are chosen in such a way that we can ensure small errors. Some numerical examples showing the good approximation results will be given.

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3.11 Function approximation and machine learning

Hrushikesh N. Mhaskar (Claremont Graduate University, US)

Main reference Hrushikesh N. Mhaskar: "A direct approach for function approximation on data defined manifolds", Neural Networks, Vol. 132, pp. 253–268, 2020.
URL http://dx.doi.org/10.1016/j.neunet.2020.08.018

Two of the fundamental problems of machine learning are the following: (1) Given random samples from an unknown probability distribution, estimate the probability measure, (2) Given samples $\{(y_j, z_j)\}$ from an unknown probability distribution, estimate a functional relationship between z_j and y_j. We explain why classical approximation theory as it was during our childhood is not adequate to solve the problem, and explain our efforts to use ideas of approximation theory to give a direct solution to the problem of estimating the functional relationship. We demonstrate that the problem of density estimation can be considered as a dual problem.

3.12 From ESPRIT to ESPIRA: Estimation of Signal Parameters by Iterative Rational Approximation

Gerlind Plonka-Hoch (Universität Göttingen, DE)

Joint work of Nadiia Derevianko, Gerlind Plonka, Markus Petz
Main reference Nadiia Derevianko, Gerlind Plonka, Markus Petz: "From ESPRIT to ESPIRA: estimation of signal parameters by iterative rational approximation", IMA Journal of Numerical Analysis, 2022.
URL http://dx.doi.org/10.1093/imanum/drab108

We consider exponential sums of the form

$$f(t) = \sum_{j=1}^{M} \gamma_j \, \mathrm{e}^{\phi_j t} = \sum_{j=1}^{M} \gamma_j \, z_j^t,$$

where $M \in \mathbb{N}$, $\gamma_j \in \mathbb{C} \setminus \{0\}$, and $z_j = \mathrm{e}^{\phi_j} \in \mathbb{C} \setminus \{0\}$ with $\phi_j \in \mathbb{C}$ are pairwise distinct. The recovery of such exponential sums from a finite set of possibly corrupted signal samples plays an important role in many signal processing applications, see e.g. in phase retrieval, signal approximation, sparse deconvolution in nondestructive testing, model reduction in system theory, direction of arrival estimation, exponential data fitting, or reconstruction of signals with finite rate of innovation.

Often, the exponential sums occur as Fourier transforms or higher order moments of discrete measures (or streams of Diracs) of the form $\sum_{j=1}^{M} \gamma_j \, \delta(\cdot - T_j)$ with $T_j \in \mathbb{R}$, which leads to the special case that $\phi_j = \mathrm{i}T_j$ is purely complex, i.e., $|z_j| = 1$.

We introduce a new method for **E**stimation of **S**ignal **P**arameters based on **I**terative **R**ational **A**pproximation (ESPIRA) for sparse exponential sums. Our algorithm uses the AAA algorithm for rational approximation of the discrete Fourier transform of the given equidistant signal values. We show that ESPIRA can be interpreted as a matrix pencil method applied to Loewner matrices. These Loewner matrices are closely connected with the Hankel matrices which are usually employed for recovery of sparse exponential sums. Due to the construction of the Loewner matrices via an adaptive selection of index sets, the matrix pencil method is stabilized. ESPIRA achieves similar recovery results for exact data as ESPRIT and the matrix pencil method (MPM) but with less computational effort. Moreover, ESPIRA strongly outperforms ESPRIT and MPM for noisy data and for signal approximation by short exponential sums.

3.13 Interpretable Approximation of High-Dimensional Data based on ANOVA Decomposition

Daniel Potts (TU Chemnitz, DE)

Joint work of Daniel Potts, Michael Schmischke
Main reference Daniel Potts, Michael Schmischke: "Approximation of High-Dimensional Periodic Functions with Fourier-Based Methods", SIAM J. Numer. Anal., Vol. 59(5), pp. 2393–2429, 2021.
URL http://dx.doi.org/10.1137/20M1354921
Main reference Daniel Potts, Michael Schmischke: "Interpretable Approximation of High-Dimensional Data", SIAM J. Math. Data Sci., Vol. 3(4), pp. 1301–1323, 2021.
URL http://dx.doi.org/10.1137/21M1407707

We consider fast Fourier based methods for the approximation of high-dimensional multivariate functions. Our aim is to learn the support of the Fourier coefficients in the frequency domain, where only function values of scattered data available. Based on the fast Fourier transform for nonequispaced data (NFFT) we will use the ANOVA (analysis of variance) decomposition in combination with the sensitivity analysis in order to obtain interpretable results. We will couple truncated ANOVA decompositions with the NFFT and compare the new method to other approaches on publicly available benchmark datasets.

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3.14 Detection of directional higher order jump discontinuities with shearlets

Jürgen Prestin (Universität zu Lübeck, DE)

Joint work of Kevin Schober, Jürgen Prestin, Serhii A. Stasyuk
Main reference Kevin Schober, Jürgen Prestin, Serhii A. Stasyuk: "Edge detection with trigonometric polynomial shearlets", Adv. Comput. Math., Vol. 47(1), p. 17, 2021.
URL http://dx.doi.org/10.1007/s10444-020-09838-3

In this talk we consider trigonometric polynomial shearlets which are special cases of directional de la Vallée Poussin type wavelets to detect singularities along curves of periodic bivariate functions. This generalises the one-dimensional case discussed in [1], where singularities occur in single points only.

Here we provide sharp lower and upper estimates for the magnitude of inner products of the shearlets with the given function. The size of these shearlet coefficients depends not only on the distance to the curve singularity, but also on the direction of the singularity.

In the proofs we use orientation-dependent localization properties of trigonometric polynomial shearlets in the time and frequency domain, cf. [1]. We also discuss jump discontinuities in higher order directional derivatives along edges, see [2].

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3.15 Multi-exponential Functions and Continued Fractions

Tomas Sauer (Universität Passau, DE)

Main reference Tomas Sauer: "Prony's method in several variables: Symbolic solutions by universal interpolation", J. Symb. Comput., Vol. 84, pp. 95–112, 2018.
URL http://dx.doi.org/10.1016/j.jsc.2017.03.006

The recovery of multi-exponential functions of the form

$$f(x) = \sum_{k=1}^{r} f_k \, \rho_k^x,$$

from integer samples is closely related to the theory of continued fractions. The connection is, among others, made by the Hankel matrices

$$H_n := \big(f(j+k) : j,k = 0,\dots,n \big)$$

and the rational best approximants to the Laurent series

$$\lambda(x) = \sum_{k=0}^{\infty} f(k) \, x^{-k-1}, \qquad x \neq 0.$$

The talk highlights some of these connections and also discusses some possibilities for extensions to several variables.

References

1 T. Sauer, *Continued Fractions and Signal Processing.* Springer, 2021.

3.16 An Inverse Problems Framework for Magnetic Resonance Relaxometry and Macromolecular Mapping

Richard G. Spencer (US)

Joint work of Richard G. Spencer, Chuan Bi
Main reference Richard G. Spencer, Chuan Bi: "A Tutorial Introduction to Inverse Problems in Magnetic Resonance", NMR in Biomedicine, Vol. 33(12), p. e4315, 2020.
URL http://dx.doi.org/10.1002/nbm.4315

The success of conventional magnetic resonance imaging (MRI) is partly attributable to the fact that it is a Fourier technique. Data is collected in the space reciprocal to spatial coordinates, known as k-space, and then (inverse) Fourier transformed to produce an image. This reconstruction has the very attractive property of being mathematically well-conditioned, with condition number of unity, so that noise in the acquired data is necessarily transmitted to the image domain but is not magnified. Because of this, early studies performed at low magnetic field and with relatively unsophisticated radio frequency technology were able to yield useful images. Correspondingly, the quality of conventional MRI has increased roughly in proportion to improvements in acquisition SNR. However, the newer technique of MR relaxometry is not so fortunate; the reconstruction of acquired data to obtain parameter distribution functions is via a version of the inverse Laplace transform. This arises from the classically ill-posed problem of inverting the Fredholm equation of the first kind. One implication is that noise is amplified in the reconstruction process, so that brute force efforts to improve SNR rapidly reach the point of diminishing returns, and other means must be undertaken to produce useful results. For this, the inverse problems perspective has proven to be enormously fruitful. We will discuss some aspects of this formalism applicable to MR relaxometry and related experiments. Our main application is to macromolecular mapping, particularly myelin mapping in the brain, and we will show how more accurate myelin quantification permits physiological correlations to be established. Our studies have the twofold goal of improving the capacity of MR to diagnose pathology and monitor disease progression, and of developing methods of general use for inverse problems.

3.17 Super-Resolution on the Two-Dimensional Unit Sphere

Hanna Veselovska (TU München, DE)

Joint work of Frank Filbir, Kristof Schröder, Anna Veselovska
Main reference Frank Filbir, Kristof Schröder, Anna Veselovska: "Recovery of Atomic Measures on the Unit Sphere", Numerical Functional Analysis and Optimization, Vol. 43(7), pp. 755–795, Taylor & Francis, 2022.
URL http://dx.doi.org/10.1080/01630563.2022.2052319

In this talk, we discuss the problem of recovering an atomic measure on the unit 2-sphere $\mathbb{S}^2$ given finitely many moments with respect to spherical harmonics. Our analysis relies on the formulation of this problem as an optimization problem on the space of bounded Borel measures on $\mathbb{S}^2$ as it was considered by Y. de Castro & F. Gamboa [1] and E. Candés & C. Fernandez-Granda [2]. We construct a dual certificate using a kernel given in an explicit form and make a concrete analysis of the interpolation problem, which provides theoretical guarantees for the recovery of atomic measure on $\mathbb{S}^2$. Next to that, we analyze such a problem from a numerical perspective in an extensive series of experiments, using a semidefinite formulation of the optimization problem and its discretized counterpart.

This is a joint work with Frank Filbir and Kristof Schröder.

References

1 De Castro, Y., & Gamboa, F., Exact reconstruction using Beurling minimal extrapolation. *Journal of Mathematical Analysis and applications*, **395**(1), 336–354, 2012.

2 Candés E.J., Fernandez-Granda C. , Super-Resolution from Noisy Data. *J. Fourier Anal. Appl.*, **19**, 1229–1254, 2013.

3 Filbir, F., Schröder, K., & Veselovska, A., Recovery of Atomic Measures on the Unit Sphere. *Numerical Functional Analysis and Optimization*, 1–41, 2022.

3.18 Antenna Position Estimation through Sub-Sampled Exponential Analysis of Signals in the Near-Field

Rina-Mari Weideman (University of Stellenbosch, ZA)

Joint work of Ridalise Louw, Ferre Knaepkens, Annie Cuyt, Wen-shin Lee, Stefan J. Wijnholds, Dirk I. L. de Villiers, Rina-Mari Weideman
Main reference Ridalise Louw, Ferre Knaepkens, Annie Cuyt, Wen-shin Lee, Stefan J. Wijnholds, Dirk I. L. de Villiers, Rina-Mari Weideman: "Antenna Position Estimation Through Subsampled Exponential Analysis of Signals in the Near Field", in URSI Radio Science Letters, vol. 3, 2021.
URL https://doi.org/10.46620/21-0062

Antenna position estimation is an important problem in large irregular arrays where the positions might not be known very accurately from the start. We present a method using harmonically related signals transmitted from an Unmanned Aerial Vehicle (UAV), with the added advantage that the UAV can be in the near-field of the receiving antenna array. The received signal samples at a chosen reference antenna element are compared to those at every other element in the array in order to find its position. We show that the method delivers excellent results using ideal synthetic data with added noise. Furthermore, we also simulate the problem in a full-wave solver. Although the results are less accurate than when synthetic data are used, due to the effects of mutual coupling, the method still performs well, with errors smaller than 4% of the smallest transmitted wavelength. Finally, we show how our method can detect whether the cables of two antennas were accidentally switched, and how a simple mutual coupling calibration method can improve the results even further.

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